E_n-coalgebras in simplicial sets

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Abstract

In this paper, we describe a simplicial analogue of the coendomorphism operad associated to the wedge product and define coalgebras in the category of simplicial sets. We show that simplicial n-fold suspensions are coalgebras up to coherent homotopy over the Barratt-Eccles E_n–operad. We also compute an explicit model for the A_∞–operad in simplicial sets, and describe the Boardman-Vogt resolution of the Barratt–Eccles E_n-operad.

1 Introduction

The little n-discs operad $\mathbb{D}_n$ was first introduced by J. P. May in his 1972 book *The Geometry of Iterated Loop Spaces* [13], although it was foreshadowed in the work of Stasheff and Boardman-Vogt. He had noticed that n-fold loop spaces carry a natural monoidal (up to homotopy) structure induced by concatenation of loops. He invented operads in order to capture this underlying structure without reference to the space itself. This approach proved its utility immediately, when he was able to show that any algebra over $\mathbb{D}_n$ is weakly homotopic to an n-fold loop space, a famous result known as May's recognition principle. Since then, this operad has informed much progress in algebraic topology. For example, it can be shown that the homology of the little n-discs operad is the parameterized Poisson operad Pois_n in chain complexes [3]. This immediately implies that the homology of n-fold loop spaces possesses not just the Pontryagin product induced by the concatenation of loops, but also a binary product of degree $1-n$ called the Browder bracket. The Dyer-Lashof and Kudo-Araki operations on the mod p cohomology of iterated loop spaces may be constructed by more complex considerations [4].

The principle of Eckmann-Hilton duality suggested that that iterated suspensions should possess a similarly rich theory. Moreno-Fernandez, Wierstra and author have made some steps in this direction in the [7]. One approach is, for each topological space X, to define the *coendomorphism operad* $\mathsf{CoEnd}(X)$ (Definition 2.1). A $\mathscr{P}$-coalgebra is defined to be a pair (X,ϕ) where X is a space and ϕ is an operadic morphism $\mathscr{P} \to \mathsf{CoEnd}(X)$. It turns out that an analogue of May's recognition principle holds in this setting and that the $\mathbb{D}_n$-coalgebras are, up to homotopy, precisely the n-fold suspensions.

The above theory is mainly developed in the setting of topological spaces, classically defined. However, modern homotopy theory is primarily studied via simplicial techniques. To connect these two strands, in this article, we extend the theory of E_n-coalgebras to the realm of simplicial sets. The theory does not extend as naiively as one might hope, as the wedge sum of fibrant objects is not necessarily fibrant (see the discussion at the start of Section 3.1). To circumnavigate this, we use Kan's Ex^∞ functor to define a coendomorphism operad completely internal to the category of simplicial sets (Definition 3.9) for any simplicial set X with only finitely many non-degenerate simplices, and use this to define coalgebras in the same way as in the last paragraph. We may then establish, via model theoretic arguments, that n-fold simplicial suspensions are E_n-coalgebras (Theorem 3.16).

A secondary goal of this article is to provide some concrete cofibrant E_n-operads in simplicial sets. This is important because in general because ∞-algebras over an operad $\mathscr{P}$ may be defined as ordinary algebras over a cofibrant replacement of $\mathscr{P}$.

The Boardman-Vogt resolution (Definition 2.24) is a convenient choice of cofibrant replacement functor in many model categories of operads. In the last part of the article, we compute some examples of the Boardman-Vogt resolution in the category of simplicial sets. We show that the arity

n component of the Boardman-Vogt resolution of the simplicial associative operad consists of $n!$ disjoint copies of a simplicial set analogous to an associahedron. We also give a similarly explicit description of the Boardman-Vogt resolution of the Barratt-Eccles E_n-operad.

Notation and conventions

All topological spaces are compactly generated and Hausdorff. We will refer to the monoidal category of such spaces equipped with the Kelley product as Top. The symmetric group on n letters is denoted $\mathbb{S}_n$.

The structure of this article

This paper has the following structure. First we recall some preliminaries on topological coalgebras, Kan's Ex^∞-functor and the Boardman-Vogt construction. Then in part 3, we construct a coendomorphism operad in simplicial sets. We prove that simplicial suspensions are E_n- coalgebras. Finally, we conclude by constructing some cofibrant models for the E_n-operad in simplicial sets.

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2 Preliminaries

In this section, we collect some of the prerequisites for understanding this paper. First we recall the theory of coalgebras in topological spaces. Then we describe Kan's Ex^∞-functor, a fibrant replacement functor in the Quillen model category of simplicial sets. Finally, we outline the construction of the W-construction, a cofibrant replacement in the category of Berger-Moerdijk model category of operads.

2.1 Coalgebras in topological spaces

In a recent preprint [7, Definition 2.14], the authors show that one can define a *coalgebra over an operad* in topological spaces. There is a similar, but not equivalent, notion in Vect given in [11, Subsection 5.2.17]. We summarise this below.

Definition 2.1. Let X be a pointed topological space. The *coendomorphism operad* $\mathsf{CoEnd}(X)$ has arity r component

$$\mathsf{CoEnd}(X)(r) := \mathrm{Map}_{\mathrm{Top}_*}(X, X^{\vee r})$$

For $r = 0$, set $\mathsf{CoEnd}(X)(0) = \mathrm{Map}_{\mathrm{Top}_*}(X, *) = *$. The operadic composition maps are defined by

$$\gamma : \mathsf{CoEnd}(X)(r) \otimes \mathsf{CoEnd}(X)(n_1) \otimes \cdots \otimes \mathsf{CoEnd}(X)(n_r) \to \mathsf{CoEnd}(X)(n_1 + \cdots + n_r)$$

$$(f, f_1, \cdots, f_n) \mapsto (f_1 \vee \cdots \vee f_n) \circ f$$

The symmetric group action permutes the wedge factors in the output.

Remark 2.2. Note that $\mathsf{CoEnd}(X)$ is naturally pointed. We will normally choose to ignore this extra structure, and will regard $\mathsf{CoEnd}(X)$ as unpointed for the rest of this report.

This immediately allows us to define a coalgebra as an algebra over the coendomorphism operad.

Definition 2.3. Let $\mathscr{P}$ be an (unpointed) operad in the category of topological spaces. A $\mathscr{P}$-*coalgebra* is a pointed space X along with an (unpointed) morphism of operads

$$\Delta : \mathscr{P} \to \mathsf{CoEnd}(X)$$

$$0 \longrightarrow (0,1) \longleftarrow 1$$

Figure 1: $\operatorname{sd}\triangle^1$

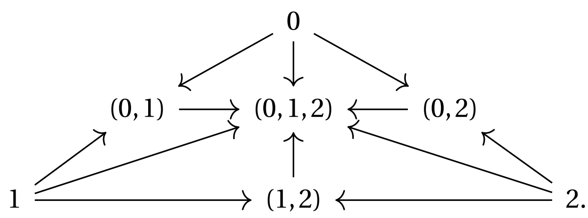


Figure 2: $\operatorname{sd}\triangle^2$

Remark 2.4. Using the product-mapping space adjunction we see that this is equivalent to giving a sequence of coproducts on the space X.

$$\Delta_r : \mathscr{P}(r) \times X \to X^{\vee r}$$

$$(\phi, x) \mapsto \Delta(\phi)(x)$$

In this framework, one can prove the following result.

Theorem 2.5. *[7, Theorem 2.1] Let $\Sigma^n X$ be the n-fold suspension of a pointed space X. Then there is a natural map of operads*

$$\Delta : \mathbb{D}_n \to \mathsf{CoEnd}(\Sigma^n X)$$

which encodes the homotopy coassociativity and homotopy cocommutativity of the pinch map. Otherwise said, n-fold suspensions are coalgebras over the little n-discs operad. Furthermore, for any based map $X \to Y$, the induced map $\Sigma^n X \to \Sigma^n Y$ extends to a morphism of $\mathbb{D}_n$-coalgebras.

2.2 Kan's Ex^∞ functor

The classical Quillen model structure on the category of simplicial sets has the property that all objects are cofibrant. On the other hand, for many computations it is useful to also have functorial fibrant replacements. This question may be resolved via Kan's Ex^∞ functor which computes them via the process of barycentric subdivision. For details of proofs we refer the reader to [8, Chapter III].

Definition 2.6. Recall that the nondegenerate simplices of the standard n-simplex $\triangle^n$ are exactly the increasing injections $[m] \to [n]$ with $0 \le m \le n$. These are in one-to-one correspondence with the subsets of $\{0,1,\dots,n\}$ of cardinality $m+1$ and thus form a poset under inclusion which we denote $P\triangle^n$. We define the *simplical subdivision* of $\triangle^n$ to be

$$\operatorname{sd}\triangle^n := \mathscr{N}(P\triangle^n)$$

where $\mathscr{N}$ is the nerve of the poset (regarded as a small category with morphisms given by inclusions).

Visually the subdivisions of the first two standard simplices are shown in Figures 1 and 2. The arrow notation $a \xrightarrow{f} b$ simply means that $d_0(f) = a$ and $d_1(f) = b$. The leading way that they have been drawn, illustrates the following lemma.

Lemma 2.7. *[8, Lemma III.4.1] On the level of geometric realizations, there is a homeomorphism $f : |\operatorname{sd}\triangle^n| \xrightarrow{\sim} |\triangle^n|$.*

The notion of subdivision can be extended to any simplicial set, not just the standard simplices. This extension makes use of the notion of a *simplex category*, which we shall introduce next.

Definition 2.8. The *simplex category* $\triangle \downarrow X$ of a simplicial set X, has for objects all simplicial maps $\sigma : \triangle^n \to X$ and has for morphisms, the commutative diagrams of the form

$$\begin{array}{ccc} \triangle^n & \xrightarrow{\sigma} & X \\ \downarrow{\theta^*} & \nearrow_{\tau} & \\ \triangle^m & & \end{array}$$

where θ^* is induced by a unique ordinal map $\theta : [m] \to [n]$.

Definition 2.9. Let X be a simplicial set. The *subdivision* $\operatorname{sd} X$ of X is defined to be the simplicial set

$$\operatorname{sd} X = \lim_{\triangle^n \to X} \operatorname{sd} \triangle^n$$

with the limit indexed by the simplex category of X.

Definition 2.10. Let X be a simplicial set. There is a natural map $\nu_{\triangle^n} : \operatorname{sd} \triangle^n \to \triangle^n$ induced by the map of posets $P\triangle^n \to [n]$ given by

$$[v_0, v_1, \ldots, v_k] \mapsto v_k.$$

The *last vertex map* $\nu_X : \operatorname{sd} X \to X$ is

$$\nu_X = \lim_{\triangle^n \to X} \nu_{\triangle^n}$$

with the limit indexed by the simplex category of X.

We define the Ex functor to be the right adjoint of the sd functor.

Definition 2.11. For any simplicial set X we define

$$\operatorname{Ex}(X)_n := \operatorname{Hom}_{\mathrm{Set}_\triangle}(\operatorname{sd} \triangle^n, X)$$

Definition 2.12. We have a morphism $\mu_X : X \to \operatorname{Ex}(X)$ which is adjoint to the last vertex map. Thus we obtain a diagram

$$X \longrightarrow \operatorname{Ex}(X) \longrightarrow \operatorname{Ex}^2(X) \longrightarrow \cdots$$

The colimit of this diagram is denoted $\operatorname{Ex}^\infty(X)$.

Example 2.13. If X is the one point simplicial set, then $\operatorname{Ex}^\infty(X)$ is isomorphic to X. Too see this, observe that the one point simplicial set $*$ is the terminal object in the category of simplicial sets. Therefore $(\operatorname{Ex} *)_n = \operatorname{Hom}_{\mathrm{Set}_\triangle}(\operatorname{sd} \triangle^n, *) = *_n$. The face and degeneracy maps are obviously also uniquely determined. Thus we can conclude that $\operatorname{Ex} * = *$ and thence by induction that $\operatorname{Ex}^n * = *$. The map $\operatorname{Ex}^n * \to \operatorname{Ex}^{n+1} *$ induced by the last vertex map must be the identity by the terminality of $*$, and the limit of the diagram

$$* \longrightarrow * \longrightarrow * \longrightarrow \cdots$$

is $*$.

The following theorem lists some useful properties of the Ex^∞ functor.

Theorem 2.14. *[8, Theorem 4.8] Let X be a simplicial set. Then:*

1. Ex^∞*(X) is a Kan complex.*
2. *The canonical map* $\eta_X : X \to \operatorname{Ex}^\infty(X)$ *is an acyclic cofibration.*
3. Ex^∞ *preserves fibrations.*
4. Ex^∞ *preserves finite limits.*

It follows that $\operatorname{Ex}^\infty(X)$ is a fibrant replacement functor for X.

2.3 The Boardman–Vogt resolution

The material in this section is drawn from [2]. In the Berger-Moerdijk model structure on operads [1], a morphism of operads, over some ambient category $\mathscr{C}$, is fibrant or a weak equivalence if and only if the underlying morphism of $\mathbb{S}$-modules is fibrant or a weak equivalence. Therefore questions involving these can be dealt with using only the tools of $\mathscr{C}$. Cofibrancy is a less directly tractable property and thus we need an alternative way to understand it. This is also quite important from the perspective of explicitly constructing ∞-algebras. It turns out that for a special class of operads one can construct a cofibrant replacement functor called the *Boardman–Vogt resolution* or the *W-construction.*

As we shall see, the actual construction is extremely technical but the intuition behind it is fairly simple. Given an arbitrary pointed operad $\mathscr{P}$ that is cofibrant as an $\mathbb{S}$-module, it is shown in [2], that $\mathscr{F}_*(F_*(\mathscr{P}))$ is cofibrant as an operad, where F_* is the forgetful functor that is right adjoint to the free operad functor $\mathscr{F}_*$. The appearance of trees in what will follow is due to their appearance in the free operad construction.

We are going to combinatorially construct a factorisation of the counit of the free–forgetful adjunction. With some assumptions on the operad $\mathscr{P}$, the factorisation

$$\mathscr{F}_*(F_*(\mathscr{P})) \hookrightarrow W(\mathscr{P}) \xrightarrow{\sim} \mathscr{P}$$

will be a weak equivalence followed by a cofibration. Thus $W(\mathscr{P})$ is a cofibrant replacement of $\mathscr{P}$.

Definition 2.15. An *interval* in **C** is a factorisation $I \sqcup I \hookrightarrow H \xrightarrow{\sim} I$ of the folding map into a cofibration $(0,1)$ followed by a weak equivalence ϵ, together with an associative operation $\vee : H \otimes H \to H$ such that all of the following diagrams commute.

$$\begin{array}{ccccc} I\otimes H & \xrightarrow{0\otimes H} & H\otimes H & \xrightarrow{H\otimes 0} & H\otimes I \\ & \searrow^{\sim} & \downarrow{\vee} & \swarrow^{\sim} & \\ & & H & & \end{array}$$

$$\begin{array}{ccccccccc} & & I\otimes H & \xrightarrow{1\otimes H} & H\otimes H & \xleftarrow{H\otimes 1} & H\otimes I & & \\ & \swarrow^{I\otimes\epsilon} & \downarrow & & \downarrow{\vee} & & \downarrow & \searrow^{\epsilon\otimes I} & \\ I\otimes I & \xrightarrow{\sim} & I & \xrightarrow{1} & H & \xleftarrow{1} & I & \xleftarrow{\sim} & I\otimes I \end{array}$$

$$\begin{array}{ccc} H\otimes H & \xrightarrow{\epsilon\otimes\epsilon} & I\otimes I \\ \downarrow{\vee} & & \downarrow{\sim} \\ H & \xrightarrow{\epsilon} & I \end{array}$$

$$\begin{array}{ccc} I & \xrightarrow{0} & H \\ \downarrow{\vee} & \searrow^{id} & \downarrow{\epsilon} \\ H & \xrightarrow{\epsilon} & I \end{array}$$

Next we shall construct an n-cube $H(T)$ associated to every tree T with n internal edges. By virtue of its construction we shall have inclusion maps into this cube from $H(T')$ for every tree T' that can be made from T by contracting edges and these inclusions glue together nicely, in the sense that if T'' is a contraction of T' then the two inclusions $H(T'') \hookrightarrow H(T)$ and $H(T'') \hookrightarrow H(T') \hookrightarrow H(T)$ agree.

Definition 2.16. Let T be a planar tree with set of internal edges $E(T)$ of cardinality k. We shall assume that we have chosen a consistent convention for ordering $E(T)$. We define

$$H(T) = \bigotimes_{e\in E(T)} H.$$

Remark 2.17. Given a tree T, we can fix any ordering of internal edges at all for our consistent convention for ordering $E(T)$. The only reason for requiring it in the first place is that when we write products like $\bigotimes_{e\in E(T)} H$ where the order of factors is important, we want the i^{th} factor in the product to always corresponds to the same internal edge.

Remark 2.18. One should observe that the symmetries of T give $H(T)$ an automatic right $\mathrm{Aut}(T)$-action.

Definition 2.19. Let T be a planar tree and let D be a subset of its set of internal edges $E(T)$. We define

$$H_D(T) = \bigotimes_{e \in E(T)} H_e$$

where

$$H_e = \begin{cases} I & \text{if } e \in D. \\ H & \text{otherwise.} \end{cases}$$

We further define

$$H^-(T) = \bigcup_{D \neq \varnothing} H_D(T)$$

Remark 2.20. Let T/D be the tree obtained by contracting the edges in D. Then there is clearly a natural isomorphism

$$H_D(T) \xrightarrow{\sim} H(T/D).$$

This extends to an acyclic cofibration

$$H_D(T) \hookrightarrow H(T)$$

where we apply $0 : I \to H$ to H_e if $e \in D$ and the identity morphism otherwise. The pushout-product condition tells us that the induced map

$$H^-(T) \hookrightarrow H(T).$$

is an acyclic fibration.

The next two definitions concern arity 1 operations. Essentially these definitions allow us to ignore vertices with only one input edge during explicit computations. Informally, we just eliminate any such vertices by operadic composition.

Definition 2.21. If c is a nonempty set of unary (meaning having exactly one incoming edge) internal vertices of a tree T, there is a map

$$r_c : H(T) \to H(T/c)$$

where T/c is given by removing each vertex of c and connecting the incoming and outgoing edge. This map is given in terms of $\vee : H \otimes H \to H$ for vertices connecting two internal edges and $\epsilon : H \to I$ for vertices connecting an internal and external edge.

Definition 2.22. Let $\mathbb{T}'$ be the set consisting of pairs (T, c), where $T \in \mathbb{T}$ and c is a nonempty set of unary internal vertices of T. We recursively define a function $\overline{\mathscr{P}}_c : \mathbb{T}' \to \mathbf{C}$ as follows. We fix $\overline{\mathscr{P}}_c(I_{\mathbb{T}}) = I$ and define

$$\overline{\mathscr{P}_c}(\gamma(t_n, T_1, \dots, T_n)) = \begin{cases} I \otimes \overline{\mathscr{P}_c}(T_1) \otimes \cdots \otimes \overline{\mathscr{P}_c}(T_n) & \text{if } x \in c; \\ \mathscr{P}(n) \otimes \overline{\mathscr{P}_c}(T_1) \otimes \cdots \otimes \overline{\mathscr{P}_c}(T_n) & \text{otherwise.} \end{cases}$$

where x is the internal root vertex of t_n.

Remark 2.23. For all nonempty sets c, d of unary vertices in T with $c \subseteq d$, by the pushout-product condition the unit $u : I \to \mathscr{P}(1)$ induces a cofibration $\overline{\mathscr{P}_d}(T) \to \overline{\mathscr{P}_c}(T)$. Let us write (observing that $\overline{\mathscr{P}_\varnothing} = \overline{\mathscr{P}}(T)$)

$$\overline{\mathscr{P}_*}(T) = \bigcup_{c \neq \varnothing} \overline{\mathscr{P}_c}(T)$$

where c ranges over all the nonempty sets of unary vertices in T and the union is interpreted as the colimit over all cofibrations $\overline{\mathscr{P}_d}(T) \to \overline{\mathscr{P}_c}(T)$ for $c \subseteq d$. An application of pushout-product condition shows that the induced map $\overline{\mathscr{P}_*}(T) \to \overline{\mathscr{P}}(T)$ is an $\mathrm{Aut}(\mathscr{P})$-cofibration for every well-pointed cofibrant operad $\mathscr{P}$.

We can now put these definitions together to create the W-construction. There are two distinct parts to this. First, we shall define the $\mathbb{S}$-module structure and then we shall define the operadic composition maps.

Definition 2.24. Let H be an interval. For any operad $\mathscr{P}$ we shall construct the operad $W(H,\mathscr{P})$ as the colimit of acyclic cofibrations of $\mathbb{S}$-modules.

$$W_0(H,\mathscr{P}) \hookrightarrow W_1(H,\mathscr{P}) \hookrightarrow W_2(H,\mathscr{P}) \hookrightarrow \cdots \tag{1}$$

The reader should be somewhat confused by this because we have not yet said what $W_i(H,P)$ is. Intuitively, $W_i(H,P)$ is the dimension $i-2$ component of $W(H,P)$; the part corresponding to trees with at most i internal edges.

More formally, we construct $W_i(H,\mathscr{P})(n)$ by induction on i. For the base case, that is $i=0$, we set

$$W_0(H,\mathscr{P})(n) := \mathscr{P}(n)$$

for all $n \geq 0$.

Now suppose that $i > 0$ and that $W_{i-1}(H,\mathscr{P})$ has been defined already. We shall also suppose that that we have a canonical map

$$\alpha_S : (H(S) \otimes \overline{\mathscr{P}}(S)) \otimes_{\mathrm{Aut}(S)} I[\mathbb{S}_n] \to W_{i-1}(H,\mathscr{P})(n)$$

for each tree S with at most $i-1$ internal edges and n input edges. (Recall that $\overline{\mathscr{P}}$ is the functor that we used in the construction of the free functor $\mathscr{F}$.) When $i=0$, these maps are

- When $n=1$, $\alpha_{I_{\mathbb{T}}}$ is the unit map $I \to \mathscr{P}(1)$
- When $n>1$, α_{t_n} is the identity map $\mathscr{P}(n) \to \mathscr{P}(n)$.

Notation 2.25. For notional simplicity we shall introduce the following shorthand

$$(H\otimes\mathscr{P})^-(T) := (H^-(T)\otimes\overline{\mathscr{P}}(T)) \cup_{H^-(T)\otimes\overline{\mathscr{P}_*}(T)} (H(T)\otimes\overline{\mathscr{P}_*}(T)).$$

Remark 2.26. By the pushout-product condition the inclusion map

$$(H\otimes\mathscr{P})^-(T) \hookrightarrow H(T)\otimes\overline{\mathscr{P}}(T)$$

is a cofibration.

Our next step shall be to use the maps α_S to construct a $\mathbb{S}_n$-equivariant map

$$\alpha_T^- : (H\otimes\overline{\mathscr{P}})^-(T) \otimes_{\mathrm{Aut}(S)} I[\mathbb{S}_n] \to W_{i-1}(H,\mathscr{P})(n). \tag{2}$$

Construction 2.27 (The construction of α_T^-)**.** For every nonempty set D of internal edges define β_D as the composite

$$\begin{aligned}(H_D(T)\otimes\overline{\mathscr{P}}(T))\otimes_{\mathrm{Aut}(T)} I[\mathbb{S}_n] &\to (H(T/D)\otimes\overline{\mathscr{P}}(T/D))\otimes_{\mathrm{Aut}(T)} I[\mathbb{S}_n]\\ &\xrightarrow{\alpha_{T/D}} W_{i-1}(H,\mathscr{P})(n)\end{aligned}$$

where the first map is induced by both

- the isomorphism $H_D(T) \to H(T/D)$.
- the partial operad composition map $\overline{\mathscr{P}}(T) \to \overline{\mathscr{P}}(T/D)$.

Taking all choices of D we produce an $\mathbb{S}_n$-equivariant map

$$\beta_T : (H^-(T)\otimes\overline{\mathscr{P}}(T))\otimes_{\mathrm{Aut}(T)} I[\mathbb{S}_n] \to W_{i-1}(H,\mathscr{P})(n). \tag{3}$$

$$x \mapsto (\bigcup_{D\neq\varnothing} \beta_D)(x) \tag{4}$$

In an analogous manner, for each nonempty set c of internal unary vertices of T, the isomorphism $\overline{\mathscr{P}_c}(T) \to \overline{\mathscr{P}}(T/c)$ and the map $H(T) \to H(T/c)$ of Remark 2.21 together induce a map

$$\alpha_T^c : (H(T) \otimes \overline{\mathscr{P}_c}(T)) \otimes_{\mathrm{Aut}(T)} I[\mathbb{S}_n] \to W_{i-1}(H, \mathscr{P})(n).$$

Once again these glue together over all possible nonempty choices of c to produce an $\mathbb{S}_n$-equivariant map

$$\delta_T : (H(T) \otimes \overline{\mathscr{P}_*}(T)) \otimes_{\mathrm{Aut}(T)} I[\mathbb{S}_n] \to W_{i-1}(H, \mathscr{P})(n). \tag{5}$$

The maps β_T and δ_T glue together to give α_T^-.

Now we take the coproduct over all isomorphism classes of trees T with n input edges and i internal edges, and construct the following pushout.

$$\begin{array}{ccc} \coprod_{[T], T \in \mathbb{T}(n,i)} (H \otimes \mathscr{P})^-(T) \otimes_{\mathrm{Aut}(T)} I[\mathbb{S}_n] & \xrightarrow{\sqcup \alpha_T^-} & W_{i-1}(H, \mathscr{P})(n) \\ \downarrow & & \downarrow \\ \coprod_{[T], T \in \mathbb{T}(n,i)} (H(T) \otimes \overline{\mathscr{P}}(T)) \otimes_{\mathrm{Aut}(T)} I[\mathbb{S}_n] & \xrightarrow{\sqcup \alpha_T} & W_i(H, \mathscr{P})(n) \end{array} \tag{6}$$

This defines both $W_i(H, \mathscr{P})$ and the maps α_T. Each inclusion $W_{i-1}(H, \mathscr{P}) \to W_i(H, \mathscr{P})$ is a acyclic cofibration in the model structure on $\mathbb{S}_n$-modules, because it is the pushout of one.

Now we define $W(H, \mathscr{P})$ as in Definition 2.24 and one observes that $\mathscr{P}(n) = W_0(H, \mathscr{P}) \to W(H, \mathscr{P})(n)$ is the composition of acyclic fibrations and so is one itself.

Remark 2.28. If we work with reduced operads, the construction above is somewhat simplified. In fact we are able to ignore all trees in $\mathbb{T}$ which have unary (one input edge) vertices. This is easy to see, because if T has a unary vertex then

$$\overline{\mathscr{P}_*}(T) = \overline{\mathscr{P}}(T).$$

It follows that

$$(H \otimes \mathscr{P})^-(T) = (H(T) \otimes \overline{\mathscr{P}}(T)) \otimes_{\mathrm{Aut}(T)} I[\mathbb{S}_n].$$

The tree T thus contributes no extra structure to the pushout $W_i(H, \mathscr{P})(n)$, and so we lose nothing by omitting it.

Next we move on to describing the operad structure of $W(H, \mathscr{P})$.

Composition in $W(H, \mathscr{P})$ is induced by tree grafting in $\mathbb{T}$. Given a tree T with n input edges and n trees $T_1, \ldots, T_n$ with k_i input edges respectively, one obtains a new tree $T' = \gamma(T, T_1, \ldots, T_n)$ with $k = k_1 + \cdots + k_n$ input edges. The n input edges of T become internal edges of T'. Therefore there is a map

$$H(T) \otimes H(T_1) \otimes \cdots \otimes H(T_n) \to H(T) \otimes H(T_1) \otimes \cdots \otimes H(T_n) \otimes I^n \to H(T').$$

where the first map is the canonical one and in the second map is induced by $1 : I \to H$ on the newly created internal edges and the identity on everything else.

Notation 2.29. For notational simplicity, we shall use the subscript $\lambda[T_i]$ to represent tensoring by $- \otimes_{\mathrm{Aut}(T)} I[\mathbb{S}_n]$.

There is a map

$$\gamma : (H(T) \otimes \overline{\mathscr{P}}(T))_{\lambda[T]} \otimes \bigotimes_{i=1}^{n} (H(T_i) \otimes \overline{\mathscr{P}}(T_i))_{\lambda[T_i]} \to (H(T') \otimes \overline{\mathscr{P}}(T'))_{\lambda[T]}.$$

which is induced by composition in the free operad $\overline{\mathscr{P}}(T) \otimes \overline{\mathscr{P}}(T_1) \otimes \cdots \otimes \overline{\mathscr{P}}(T_n) \to \overline{\mathscr{P}}(T')$. The operad structure is determined by the unique requirement that the following diagram commutes

$$\begin{array}{ccc} (H(T) \otimes \overline{\mathscr{P}}(T))_{\lambda[T]} \otimes \bigotimes_{i=1}^{n} (H(T_i) \otimes \overline{\mathscr{P}}(T_i))_{\lambda[T_i]} & \longrightarrow & W(H, \mathscr{P})(n) \otimes \bigotimes_{i=1}^{n} W(H, \mathscr{P})(k_i) \\ \downarrow \gamma & & \downarrow \\ (H(T') \otimes \overline{\mathscr{P}}(T'))_{\lambda[T]} & \xrightarrow{\alpha_{T'}} & W(H, \mathscr{P})(n). \end{array}$$

Theorem 2.30. *[2, Theorem 5.1] Let* $\mathbf{C}$ *be a cofibrantly generated monoidal model category with a cofibrant unit I and an interval H. Let* $\mathscr{P}$ *be a* $\mathbb{S}$*-cofibrant well-pointed operad. Then the counit of the adjunction between pointed* $\mathbb{S}$*-modules and operads admits a factorisation*

$$\mathscr{F}_*(\mathscr{P}) \hookrightarrow W(H,\mathscr{P}) \xrightarrow{\sim} \mathscr{P}$$

into a cofibration f followed by a weak equivalence g. In particular, $W(H,\mathscr{P})$ *is a cofibrant resolution for* $\mathscr{P}$*.*

2.4 The Barratt-Eccles operad and the Smith filtration

The last preliminary that we shall need is a simplicial model for E_n-operads. This is provided by the Barratt-Eccles operad.

Definition 2.31. The simplicial sets defining the Barratt-Eccles operad in each arity are of the form

$$\Gamma(r)_n = \{(w_0,\dots,w_n) \in \mathbb{S}_r \times \cdots \times \mathbb{S}_r\}$$

equipped with face and degeneracy maps

$$d_i(w_0,\dots,w_n) = (w_0,\dots,w_{i-1},\hat{w}_i,w_{i+1},\dots,w_n)$$
$$s_i(w_0,\dots,w_n) = (w_0,\dots,w_{i-1},w_i,w_i,w_{i+1},\dots,w_n).$$

$\mathbb{S}_r$ acts on $\Gamma(n)$ diagonally, that is to say if $\sigma \in \mathbb{S}_n$ and $(w_0,\dots,w_n) \in \Gamma(n)$ the

$$(w_0,\dots,w_n) * \sigma = (w_0 * \sigma,\dots,w_n * \sigma)$$

Finally the compositions are also defined componentwise via the explicit composition law of

$$\gamma : \mathbb{S}(r) \otimes \mathbb{S}(n_1) \otimes \cdots \otimes \mathbb{S}(n_r) \to \mathbb{S}(n_1 + \cdots + n_r)$$
$$(\sigma,\sigma_1,\dots,\sigma_r) \mapsto \sigma_{n_1\cdots n_r} \circ (\sigma_1 \times \cdots \times \sigma_r)$$

where $\sigma_{n_1\cdots n_r}$ is the permutation that acts on $n_1 + \cdots + n_r$ elements, by dividing them into r blocks, the first of length n_1, the second of length n_2 and so on. It then rearranges the blocks according to σ, maintaining the order within each block.

Definition 2.32. Let k be a positive integer. The *Barratt-Eccles* E_k*-operad* is defined by

$$\Gamma^{(k)}(n) = \{x \in \Gamma(n) : \theta^*_{ij}(x) \in \mathrm{sk}_{k-1}\Gamma(2) \text{ for all } i < j\}.$$

where sk_{k-1} denotes $(k-1)$-skeleton and the notation $\theta^*_{ij} : \Gamma(n) \to \Gamma(2)$, for $0 < i < j \le n$, is the coordinate-wise extension of the map $\theta_{ij}\mathbb{S}_n \to \mathbb{S}_2$ given by

$$\theta_{ij}\sigma = \begin{cases} \mathrm{id} & \text{if } \sigma(i) < \sigma(j). \\ \tau & \text{otherwise.} \end{cases}$$

The $\Gamma^{(k)}$ defines a filtration of operads of Γ, called the *Smith filtration*

$$\Gamma^{(1)} \hookrightarrow \Gamma^{(2)} \hookrightarrow \cdots \hookrightarrow \Gamma^{(i)} \cdots \hookrightarrow \Gamma^{(\infty)} = \Gamma$$

corresponding to the usual filtration

$$\mathbb{D}_1 \hookrightarrow \mathbb{D}_2 \hookrightarrow \cdots \hookrightarrow \mathbb{D}_i \cdots \hookrightarrow \mathbb{D}_\infty = \Gamma$$

given by the equatorial embeddings.

3 Coalgebras in simplicial sets

The purpose of this section is to generalise [7, Proposition 2.23] , which says that n-fold suspensions are coalgebras over the little n-discs operad, to simplicial sets. There are numerous reasons why this is a useful idea. Our initial motivation was to find a strictly coassociative topological coalgebra, to provide the Eckmann-Hilton dual of the Moore loop space. Unfortunately this turns out not to be possible, see [7, Remark 2.19]. The main motivation therefore is that passing through simplicial sets is a necessary step on the journey to describing E_n-coalgebras in the modern formalism of ∞-operads [12].

Our discussion shall proceed as follows. In the first section, we shall shall define the 'correct' notion of a coendomorphism operad in topological spaces. In the second section we will show that simplicial n-fold suspensions are homotopy algebras over the Barratt-Eccles operad, the main result of this thesis.

3.1 The simplicial coendomorphism operad

In this section, we wish to extend the notion of coalgebras to the realm of simplicial sets. As in topological spaces [7], we are going to do this by defining the notion of a coendomorphism operad. This is significantly more difficult than it appears. We cannot simply take the obvious choice, the operad defined in arity n by

$$\mathrm{Map}_{\mathrm{Set}_\triangle}(X, X^{\vee n}).$$

To see why, consider what happens in the case where $X = S^1$. This simplicial set has only 1 non-degenerate simplex other than the base point - the 1-simplex σ. The vertices of the simplicial set

$$\mathrm{Map}_{\mathrm{Set}_\triangle}(S^1, (S^1)^{\vee n})$$

are distinctly non-interesting, because the σ can only mapped to one copy of S^1 in the wedge product. This is in total contrast with the interesting structure in topological spaces, and hence in the homotopy category.

This hints at the underlying problem. As we have seen throughout this report, not all simplicial sets are fibrant in the Kan-Quillen model structure. Thus, not all maps in the homotopy category exist between all pairs of objects in the model. To ensure that they do we must take a fibrant replacement of $X^{\vee n}$. (We shall later see that $\mathrm{Map}_{\mathrm{Set}_\triangle}(X, X^{\vee n})$ is not even the correct homotopy type to be a good candidate for a coendomorphism operad.) To ensure things remain as combinatorially tractable as possible, we shall use Kan's Ex^∞ functor for this task (we could alternatively use $S_\bullet|X|$, and we shall actually use this approach in the proof of Theorem 3.16). The underlying $\mathbb{S}$-module of the desired operad is very easy to describe and we can do this immediately.

Definition 3.1. We define the *simplicial coendomorphism* $\mathbb{S}$*-module* in arity r to be

$$\mathrm{CoEnd}(X)(r) := \mathrm{Map}_{\mathrm{Set}_\triangle}(X, \mathrm{Ex}^\infty(X^{\vee r})).$$

Each $\sigma \in \mathbb{S}_r$ induces a map $\sigma^* : X^{\vee r} \to X^{\vee r}$, by via permutation of the factors of the wedge product. Then the symmetric action of the $\mathbb{S}$-module is given by the maps

$$- * \sigma : \mathrm{CoEnd}(X)(r) \to \mathrm{CoEnd}(X)(r)$$

$$f \mapsto \sigma^* \circ f.$$

Remark 3.2. It is obvious that $- * \sigma$ is a bona fide simplicial map because the degeneracy and face maps of the simplicial mapping space act only on the domain of a n-simplex $f : X \times \triangle^m \to \mathrm{Ex}^\infty(X^{\vee r})$ and not on the codomain.

The next few pages consist of defining the operadic composition maps. We start by recalling some notation.

Observation 3.3. Recall from Section 1 that $\mathrm{Ex}^{\infty}(X)$ is defined the colimit of the following chain of acyclic cofibrations

$$X \xrightarrow{\sim} \mathrm{Ex}(X) \xrightarrow{\sim} \mathrm{Ex}^2(X) \xrightarrow{\sim} \cdots \xrightarrow{\sim} \mathrm{Ex}^i(X) \xrightarrow{\sim} \cdots$$

Since cofibrations are injective in the Kan-Quillen model structure, this means that for all $x \in \mathrm{Ex}^{\infty}(X)$ there exists an $N > 0$ such that $x \in \mathrm{Ex}^n(X)$ for all $n > N$. Of course, we are implicitly identifying each $\mathrm{Ex}^n(X)$ with its image in $\mathrm{Ex}^{\infty}(X)$, where they form an exhaustive filtration.

Definition 3.4. Let X be a simplicial set with only finitely many non-degenerate simplices, and let f be an n-simplex of $\mathrm{CoEnd}(X)(r)$. In other words,

$$f \in \mathrm{Map}_{\mathrm{Set}_{\triangle}}(X, \mathrm{Ex}^{\infty}(X^{\vee r}))_n.$$

By the definition of simplicial mapping sets, f is a simplicial function $X \times \triangle^m \to \mathrm{Ex}^{\infty}(X^{\vee r})$. Following Observation 3.3, we can associate an integer N_σ to every simplex $\sigma \in X \times \triangle^m$; this being the smallest N such that $f(\sigma) \in \mathrm{Ex}^N(X^{\vee r})$. We define N_f to be the integer $\max\{N_\sigma\}_{\sigma \in X \times \triangle^m}$.

Remark 3.5. The integer N_f is well-defined because $X \times \triangle^m$, the domain of f, has only finitely many non-degenerate simplices.

Remark 3.6. It is easy to scheck the following three properties of N_f.

- f factors through $\mathrm{Ex}^{N_f}(X^{\vee r})$.
- N_f is the smallest integer with this property.
- For all $N \geq N_f$, f factors through $\mathrm{Ex}^N(X^{\vee r})$.

Our definition of the coendomorphism operad will make heavy use of the adjunction between Ex and sd. For ease of reading, we shall introduce two pieces of helpful notation.

Notation 3.7. Let $f \in \mathrm{Hom}_{\mathrm{Set}_{\triangle}}(\mathrm{sd}^N(X \times \triangle^m), (X^{\vee r}))$ for $N > 0$. This is adjoint to $f^c \in \mathrm{Hom}_{\mathrm{Set}_{\triangle}}((X \times \triangle^m), \mathrm{Ex}^m(X^{\vee r}))$. Now f^c uniquely extends to an element of $\mathrm{Hom}_{\mathrm{Set}_{\triangle}}((X \times \triangle^m), \mathrm{Ex}^{\infty}(X^{\vee r}))$ which is the same thing as $\mathrm{Map}_{\mathrm{Set}_{\triangle}}(X, \mathrm{Ex}^{\infty}(X^{\vee r}))_n$. We shall denote this element as $\overline{f}$.

Notation 3.8. Let $f \in \mathrm{Map}_{\mathrm{Set}_{\triangle}}(X, \mathrm{Ex}^{\infty}(X^{\vee r}))_m$. Then it follows from Remark 3.5 that for all $N \geq N_f$, there is a unique element, which we shall denote (f, N), of $\mathrm{Hom}_{\mathrm{Set}_{\triangle}}(\mathrm{sd}^N(X \times \triangle^m), X^{\vee r})$, such that $\overline{(f, N)} = f$.

Having dispensed with the preliminaries we are now in a position to define the composition maps. Observe that the subdivision functor is a left adjoint and so preserves colimits. In particular, it commutes with wedge products.

Definition 3.9. Let $f \in \mathrm{CoEnd}(X)(r)_m$ and $f_i \in \mathrm{CoEnd}(X)(n_i)_m$ for $1 \leq i \leq r$. We define the composition map

$$\gamma : \mathrm{CoEnd}(X)(r) \times \mathrm{CoEnd}(X)(n_1) \times \cdots \mathrm{CoEnd}(X)(n_r) \to \mathrm{CoEnd}(X)(n_1 + \cdots + n_r)$$

to be $\overline{F}$ where F is the map

$$\begin{aligned} F : \mathrm{sd}^{N+N_f}(X \times \triangle^m) &\xrightarrow{\mathrm{sd}^N(\delta_{\triangle^m})} \mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \mathrm{sd}^{N_f}(X \times \triangle^m)) \\ &\xrightarrow{\mathrm{sd}^N(\mathrm{id} \times \mathrm{sd}^{N_f}(\pi_2))} \mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \mathrm{sd}^{N_f}(\triangle^m)) \xrightarrow{a} \mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \triangle^m) \\ &\xrightarrow{(f, N_f)} \mathrm{sd}^N(X^{\vee r} \times \triangle^m)) \xrightarrow{b} \mathrm{sd}^N(X \times \triangle^m)^{\vee r} \xrightarrow{(f_1, N) \vee \cdots \vee (f_r, N)} X^{\vee n_1 + \cdots n_r} \end{aligned}$$

where

- N is the integer $\max(N_{f_1}, \ldots, N_{f_n})$.
- $\delta_{\mathrm{sd}^{N_f}(X \times \triangle^m)} : \mathrm{sd}^{N_f}(X \times \triangle^m) \to \mathrm{sd}^{N_f}(X \times \triangle^m) \times \mathrm{sd}^{N_f}(X \times \triangle^m)$ is the diagonal map.
- $\pi_2 : X \times \triangle^m \to \triangle^m$ is the projection.

- $a : \mathrm{sd}^{N_f}(\triangle^m)) \to \mathrm{sd}^N(\mathrm{sd}^{N_f}(X\times\triangle^m)\times\triangle^m)$ is the map $\mathrm{sd}^N(\mathrm{id}\times\nu^{(N_f)}_{\triangle^m})$ where $\nu^{(N_f)}_{\triangle^m} := \nu_{\triangle^m}\circ\cdots\circ\nu_{\mathrm{sd}^{N_f-1}\triangle^m}$ and $\nu_Z : \mathrm{sd}\, Z \to Z$ is the last vertex map.
- b is an isomorphism, as $\times$ is distributive over the wedge product, and the wedge product commutes with subdivision.

We need to check that the definition above gives rise to well-defined operad. We phrase this result as a theorem.

Theorem 3.10. *Let X be a simplicial set with finitely many non-degenerate simplices. Then the composition maps of Definition 3.9 induce an operad structure on the $\mathbb{S}$-module* $\mathsf{CoEnd}(X)$.

Before proving this theorem, we wish to make two useful remarks and introduce a final piece of notation.

Remark 3.11. Our first remark concerns the relationship between (f,N) and (f,M) for $M > N \geq N_f$. From the definition of Ex we see that, for all simplicial sets Z and Z', the simplicial morphism $\mathrm{Hom}_{\mathrm{Set}_\triangle}(\nu_Z, Z')$ is adjoint to $\mathrm{Hom}_{\mathrm{Set}_\triangle}(Z,\mu_{Z'})$, where both

$$\mu_Z : Z \to \mathrm{Ex}(Z).$$

$$\nu_Z : \mathrm{sd}\, Z \to Z.$$

are the maps induced by the last vertex map. Thus we have the relation

$$(f,N)\circ\nu_{\mathrm{sd}^N(X\times\triangle^m)} = (f,N+1).$$

for all $N \geq N_f$ and its obvious extension by induction. A second useful well-known result about ν_Z that you should keep in the back of your mind is that the following diagram commutes

$$\begin{array}{ccc} \mathrm{sd}\, Z & \xrightarrow{\nu_Z} & Z \\ \downarrow{\scriptstyle \mathrm{sd} f} & & \downarrow{\scriptstyle f} \\ \mathrm{sd}\, Z' & \xrightarrow{\nu_{Z'}} & Z'. \end{array} \tag{7}$$

Notation 3.12. We define $\nu^{(k)}_Z := \nu_Z\circ\cdots\circ\nu_{\mathrm{sd}^{k-1}Z}$.

Remark 3.13. Another useful thing is to note that we can replace N_f in the definition of F with any integer $K \geq N_f$, and F will not change. To see why, call this new map $F(K)$, and then observe, with the help of Diagram 7, that $F(K) = F\circ\nu^{(K-N_f)}_{\mathrm{sd}^{N_f}(X\times\triangle^m)}$. By our previous remark

$$\overline{F\circ\nu^{(K-N_f)}_{\mathrm{sd}^{N_f}(X\times\triangle^m)}} = \overline{F}.$$

Similarly, if we replace N in the definition with a larger integer K', the function F in Definition 3.9 will become another function which we will call $F(K')$. It once again follows from Remark 3.11 and Diagram 7 that this function will be related to F by the identity

$$f(K') = F\circ\nu^{(K'-N)}_Z,$$

and so we can also replace N with any larger integer in Definition 3.9 without changing the operad structure.

Theorem 3.10. We need to verify that this defines an operad, starting with the associativity axiom. So we wish to show that

$$\gamma(\gamma(f,f_1,\ldots,f_r),f_{1,1},\ldots,f_{r,n_r}) = \gamma(f,\gamma(f_1,f_{1,1},\ldots f_{1,n_1}),\ldots,\gamma(f_1,f_{r,1},\ldots f_{r,n_r}))$$

for all $f\in \mathrm{CoEnd}(X)(r)_m$, $f_i\in\mathrm{CoEnd}(X)(n_i)_m$ and $f_{i,j}\in\mathrm{CoEnd}(X)(n_{i,j})_m$. Expanding the left hand side of this we obtain

$$(\bigvee_{i=1}^{r}\bigvee_{j=1}^{r_i}(f_{ij},M))\circ(\bigvee_{k=1}^{r}\mathrm{sd}^M((f_k,M')\times\nu^{(M')}_{\triangle^m}\circ\mathrm{sd}^{M'}(\pi_2)))\circ\mathrm{sd}^{M+M'}((f,N_f)\times(\nu^{(N_f)}_{\triangle^m}\circ\mathrm{sd}^{N_f}(\pi_2))) \tag{8}$$

where $M = \max\{N_{f_{ij}}\}_{1\geq i\geq r, 1\leq j\leq r_i}$ and $M' = \max\{N_{f_i}\}_{1\leq i\leq r}$. Now let $M_i = \max\{M_{ij}\}_{0\leq j\leq r_i}$ and recall that

$$(f, M) = (f, M_i) \circ \nu^{(M-M_i)}_{\mathrm{sd}^M(X\times\triangle^m)}$$

We may deduce from this that Expression (8) can be written

$$(\bigvee_{i=1}^{r}\bigvee_{j=1}^{r_i}(f_{ij}, M_i)\circ\nu^{(M-M_i)}_{\mathrm{sd}^{M_i}(X\times\triangle^m)})\circ(\bigvee_{k=1}^{r}\mathrm{sd}^M((f_k, M')\times\nu^{(M')}_{\triangle^m}\circ\mathrm{sd}^{M'}(\pi_2)))$$
$$\circ\,\mathrm{sd}^{M+M'}((f, N_f)\times(\nu^{(N_f)}_{\triangle^m}\circ\mathrm{sd}^{N_f}(\pi_2))).$$

This can be written

$$(\bigvee_{i=1}^{r}\bigvee_{j=1}^{r_i}(f_{ij}, M_i))\circ(\bigvee_{k=1}^{r}\nu^{(M-M_k)}_{\mathrm{sd}^M(X^{\vee r_k}\times\triangle^m)}\circ\mathrm{sd}^M((f_k, M')\times\nu^{(M')}_{\triangle^m}\circ\mathrm{sd}^{M'}(\pi_2)))$$
$$\circ\,\mathrm{sd}^{M+M'}((f, N_f)\times(\nu^{(N_f)}_{\triangle^m}\circ\mathrm{sd}^{N_f}(\pi_2))).$$

Using the commutativity of Diagram 7 we see that this is equal to

$$(\bigvee_{i=1}^{r}\bigvee_{j=1}^{r_i}(f_{ij}, M_i))\circ(\bigvee_{k=1}^{r}\mathrm{sd}^{M_k}((f_k, M')\times\nu^{(M')}_{\triangle^m}\circ\mathrm{sd}^{M'}(\pi_2))\circ\nu^{M-M_k}_{\mathrm{sd}^{M'+M_k}(X\times\triangle^m)})$$
$$\circ\,\mathrm{sd}^{M+M'}((f, N_f)\times(\nu^{(N_f)}_{\triangle^m}\circ\mathrm{sd}^{N_f}(\pi_2))).$$

Once again using Diagram 7, we can rewrite this as

$$(\bigvee_{i=1}^{r}(\bigvee_{j=1}^{r_i}(f_{ij}, M_i))\circ\mathrm{sd}^{M_i}((f_i, M_{f_i})\times(\nu^{(M_{f_i})}_{\triangle^m}\circ\mathrm{sd}^{M_{f_i}}(\pi_2)))\circ\nu^{M+M'-M_i-M_{f_i}}_{\mathrm{sd}^{M'+M_i}(X\times\triangle^m)})$$
$$\circ\,\mathrm{sd}^{M+M'}((f, N_f)\times(\nu^{(N_f)}_{\triangle^m}\circ\mathrm{sd}^{N_f}(\pi_2))).$$

The above expression is equal to

$$\bigvee_{i=1}^{r}(\overline{\gamma(f_i, f_{i1},\ldots,f_{ir_i})}, M+M')\circ\mathrm{sd}^{M+M'}((f, N_f)\times(\nu^{(N_f)}_{\mathrm{sd}^{M+M'}(\triangle^m)}\circ\mathrm{sd}^{N_f}(\pi_2))).$$

By our argument on the last page, this is equal to

$$\gamma(f, \gamma(f_1, f_{1,1},\ldots f_{1,n_1})\ldots,\gamma(f_1, f_{r,1},\ldots f_{r,n_r}))$$

as desired.

The identity element of the operad is $\mu_X : X \to \mathrm{Ex}^\infty(X)$. Verifying the equivariance axioms is straightforward, it is almost exactly the same as verifying them for the topological coendomorphism operad. Therefore we have defined an operad. □

It remains only to define simplicial coalgebras, which proceeds exactly as one would expect.

Definition 3.14. Let $\mathscr{P}$ be an operad in simplicial sets. We shall say that a finite simplicial set X is a *$\mathscr{P}$-coalgebra* if there exists an operadic morphism $\Phi : \mathscr{P} \to \mathsf{CoEnd}(X)$.

Lastly we define E_n-algebras in $\mathrm{Set}_\triangle$.

Definition 3.15. In simplicial sets, an *E_n-coalgebra* is a coalgebra over the W-construction of the Barratt-Eccles E_n-operad.

3.2 Simplicial suspensions are E_n-coalgebras

In this section, in direct analogy with Moreno-Fernández and Wierstra's result in topological spaces, we aim to show that simplicial suspensions are E_n-coalgebras. The strategy of this proof is as follows. First we transfer the little n-discs operad $\mathbb{D}_n$, the topological coendomorphism operad and the operad morphism between them Φ into the category of simplicial sets using the simplicial chains functor $S_\bullet$. We then use the homotopy transfer principle to lift this to a morphism from a cofibrant replacement of $\mathbb{D}_n$ to the simplicial coendomorphism operad.

The precise statement of the simplicial version of Theorem 2.5 is as follows.

Theorem 3.16. *Let $n \in \mathbb{N}$ and $\Sigma^n X$ be the n–fold suspension of a finite simplicial set X. Then $\Sigma^n X$ has the structure of an E_n-coalgebra.*

Our proof of this theorem requires that the Cartesian product commutes with the geometric realization functor. This is actually not true in general. Therefore, we shall need to restrict from the category of all topological spaces to the category of compactly generated Hausdorff spaces and we take our product to be the Kelley product.

We also wish to be able to transfer operads from topological space to simplicial sets. This is made possible by the following definition.

Definition 3.17. Let $\mathscr{P}$ be an operad in Top. We define an operad $S_\bullet\mathscr{P}$ over $\mathrm{Set}_\triangle$ with arity n component

$$(S_\bullet\mathscr{P})(n) := S_\bullet(\mathscr{P}(n))$$

where $S_\bullet$ is the singular chains functor. The action of $\sigma \in \mathbb{S}_n$ on $S_\bullet\mathscr{P}(n)$ is given by $S_\bullet\mathscr{P}(n) * \sigma := S_\bullet(\mathscr{P}(n) * \sigma)$. The operadic composition map is $\gamma_{S_\bullet\mathscr{P}} := S_\bullet(\gamma_{\mathscr{P}})$ and we take the unit to be the simplex $[\triangle^0 \to 1_{\mathsf{Top}}] \in S_\bullet\mathscr{P}(1)$.

Remark 3.18. The operad composition map in the definition above is well-defined because $S_\bullet$ is right adjoint to the geometric realization. This means that it preserves limits, and in particular, products.

We can actually define $S_\bullet(\mathsf{CoEnd}_{\mathsf{Top}}(|X|))$ to be an alternative coendomorphism operad. The following theorem gives us a precise description of it.

Lemma 3.19. *Let X be a simplicial set with only finitely many nondegenerate simplices. The operad $S_\bullet(\mathsf{CoEnd}_{\mathsf{Top}}(|X|))$ is isomorphic to the simplicial operad $Q(X)$ with arity r component equal to*

$$Q(X)(r) := \mathrm{Map}_{\mathrm{Set}_\triangle}(X, S_\bullet|X^{\vee r}|).$$

Let $f \in Q(X)(r)_m$ and let $f_i \in Q(X)(n_i)_m$ for $1 \le i \le r$. The operadic composition map

$$\gamma : Q(X)(r) \times Q(X)(n_1) \times \cdots \times Q(X)(n_r) \to Q(X)(n_1 + \cdots + n_r)$$

is given by the adjoint under the Top-$\mathrm{Set}_\triangle$ *adjunction of $F : |X \times \triangle^m| \to |X^{\vee n_1+\cdots+n_r}|$, where F is defined by*

$$|X \times \triangle^m| \xrightarrow{|\mathrm{id}\times\delta_{\triangle^m}|} |X \times \triangle^m \times \triangle^m| \xrightarrow{a} |X \times \triangle^m| \times |\triangle^m| \xrightarrow{|f|\times\mathrm{id}} |S_\bullet|X^{\vee r}|| \times |\triangle^m| \xrightarrow{\epsilon_{X^{\vee r}}\times\mathrm{id}}$$
$$|X^{\vee r}| \times |\triangle^m| \xrightarrow{b} |X \times \triangle^m|^{\vee r} \xrightarrow{\bigvee_{i=1}^r |f_i|} \bigvee_{i=1}^r |S_\bullet|X^{\vee n_i}|| \xrightarrow{\bigvee_{i=1}^r \epsilon_{X^{\vee n_i}}} \bigvee_{i=1}^r |X^{\vee n_i}| \xrightarrow{c} |X^{\vee n_1+\cdots+n_r}|$$

where

- *$\delta_{\triangle^m} : \triangle^m \to \triangle^m \times \triangle^m$ is the diagonal map.*
- *for Y a topological space, the map $\epsilon_Y : |S_\bullet(Y)| \to Y$ is the counit of the adjunction between topological spaces and simplicial sets.*
- *$a : |X \times \triangle^m \times \triangle^m| \to |X \times \triangle^m| \times |\triangle^m|$ is an isomorphism, as $\times$ commutes with geometric realisation.*
- *$b : |X^{\vee r}| \times |\triangle^m| \to |X \times \triangle^m|^{\vee r}$ is an isomorphism, as both $\times$ and the wedge product commute with geometric realisation.*
- *$c : \bigvee_{i=1}^r |X^{\vee n_i}| \to |X^{\vee n_1+\cdots+n_r}|$ is an isomorphism, as the wedge product commutes with geometric realisation.*

For each $\sigma \in \mathbb{S}_r$, there is a map $\sigma^* : X^\vee \to X^\vee$ given by permuting the terms of the wedge sum by σ. The symmetric structure on $Q(X)(r)$ is defined by post-composition with the morphism $S_\bullet|\sigma^*|$.

Proof. We can write

$$S_\bullet(\mathsf{CoEnd}_{\mathsf{Top}}(|X|))(r) = S_\bullet\,\mathrm{Map}_{\mathsf{Top}}(|X|, |X^{\vee r}|) \cong \mathrm{Map}_{\mathrm{Set}_\triangle}(X, S_\bullet|X^{\vee k}|).$$

because, for all $K \in \mathrm{Set}_\triangle$ and $Y \in \mathsf{Top}$, we have

$$\mathrm{Hom}_{\mathsf{Top}}(|\triangle^m|, \mathrm{Map}_{\mathsf{Top}}(|K|, Y)) \cong \mathrm{Hom}_{\mathsf{Top}}(|\triangle^m| \times |K|, Y)$$

by tensor-hom adjunction. Here it is critical to distinguish between the simplicial mapping space and the hom-set. We then have

$$\mathrm{Hom}_{\mathsf{Top}}(|\triangle^m| \times |K|, Y) \cong \mathrm{Hom}_{\mathsf{Top}}(|\triangle^m \times K|, Y)$$

by the identity $|X| \times |Y| \cong |X \times Y|$ and finally we have

$$\mathrm{Hom}_{\mathsf{Top}}(|\triangle^m \times K|, Y) \cong \mathrm{Hom}_{\mathrm{Set}_\triangle}(\triangle^m \times K, S_\bullet Y)$$

by adjunction.

Secondly, it remains to check that operad morphisms are as described in the statement of the lemma. We can describe the induced operad structure on $\mathrm{Hom}_{\mathsf{Top}}(|\triangle^m \times X|, |X^{\vee r}|)$ quite easily. For $f \in \mathrm{Hom}_{\mathsf{Top}}(|\triangle^m \times X|, |X^{\vee r}|)$ and $f_i \in \mathrm{Hom}_{\mathsf{Top}}(|\triangle^m \times X|, |X^{\vee n_i}|)$ the composite $\gamma(f, f_1, \ldots f_n)$ is the function

$$F : |X \times \triangle^m| \xrightarrow{|\mathrm{id} \times \delta_{\triangle^m}|} |X \times \triangle^m \times \triangle^m| \xrightarrow{a} |X \times \triangle^m| \times |\triangle^m| \xrightarrow{f \times \mathrm{id}}$$
$$|X^{\vee r}| \times |\triangle^m| \xrightarrow{b} |X \times \triangle^m|^{\vee r} \xrightarrow{\bigvee_{i=1}^r f_i} \bigvee_{i=1}^{r} |X^{\vee n_i}| \xrightarrow{c} |X^{\vee n_1 + \cdots + n_r}|$$

The isomorphism

$$G : \mathrm{Hom}_{\mathrm{Set}_\triangle}(\triangle^m \times X, S_\bullet |X^{\vee r}|) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Top}}(|\triangle^m \times X|, |X^{\vee r}|)$$

can be written by

$$f \mapsto \epsilon_{X^{\vee r}} \circ |f|.$$

Therefore the composition map is exactly as described. □

Remark 3.20. It is important to remark that this operad will usually have a different homotopy type to the naïve simplicial coendomorphism operad mentioned at the start of this section.

Despite Remark 3.20, the simplicial coendomorphism operad and the operad $S_\bullet(\mathsf{CoEnd}_{\mathsf{Top}}(|X|))$ **will** be equivalent.

Lemma 3.21. *Let X be a finite simplicial set. Then the simplicial coendomorphism operad and the operad $S_\bullet(\mathsf{CoEnd}_{\mathsf{Top}}(|X|))$ are weakly equivalent.*

We shall prove this by constructing a zig-zig involving a third operad, which we define will first.

Definition 3.22. Let X be a finite simplicial set. Then the *mixed coendomorphism operad* $R(X)$ has arity r component

$$R(X)(r) = \mathrm{Map}_{\mathrm{Set}_\triangle}(X, \mathrm{Ex}^\infty(S_\bullet |X^{\vee r}|)).$$

For each $\sigma \in \mathbb{S}_r$, there is a map $\sigma^* : X^\vee \to X^\vee$ given by permuting the terms of the wedge sum by σ. The symmetric structure on $R(X)(r)$ is defined by post-composition with the morphism $\mathrm{Ex}^\infty(S_\bullet|\sigma^*|)$. We shall define the operadic composition map using both the sd–Ex and the simplicial chains-geometric realization adjunctions consecutively. Let $f \in Q(X)(r)_m$ and $f_i \in Q(X)(n_i)_m$ for $1 \leq i \leq r$, then the operadic composition map

$$\gamma : R(X)(r) \times R(X)(n_1) \times \cdots \times R(X)(n_r) \to R(X)(n_1 + \cdots + n_r)$$

is defined to be $\overline{F}$ which is adjoint, under the sd-Ex adjunction, of the morphism, $F : \mathrm{sd}^{N_f}(X \times \triangle^m) \to S_\bullet |X^{\vee n_1 + \cdots + n_r}|$. F is itself an adjoint, this time under the geometric realization –simplicial chains adjunction, of a morphism $G\colon|\mathrm{sd}^{N+N_f}(X\times\triangle^m)|\to|X^{\vee n_1+\cdots+n_r}|$ which we define to be the composite

$$\begin{aligned}
&|\mathrm{sd}^{N+N_f}(X\times\triangle^m)| \xrightarrow{|\mathrm{sd}^N(\delta_{\triangle^m})|} |\mathrm{sd}^N(\mathrm{sd}^{N_f}(X\times\triangle^m)\times\mathrm{sd}^{N_f}(X\times\triangle^m))|\\
&\xrightarrow{|\mathrm{sd}^N(\mathrm{id}\times\mathrm{sd}^{N_f}(\pi_2))|} |\mathrm{sd}^N(\mathrm{sd}^{N_f}(X\times\triangle^m)\times\mathrm{sd}^{N_f}(\triangle^m))| \xrightarrow{a} |\mathrm{sd}^N(\mathrm{sd}^{N_f}(X\times\triangle^m)\times\triangle^m)|\\
&\xrightarrow{\mathrm{sd}^N((f,N_f)\times\mathrm{id})} |\mathrm{sd}^N(S_\bullet|X^{\vee r}|\times\triangle^m)| \xrightarrow{b} |S_\bullet|X^{\vee r}|\times\triangle^m| \xrightarrow{c} |X^{\vee r}\times\triangle^m|\\
&\xrightarrow{d} |\mathrm{sd}^N(X^{\vee r}\times\triangle^m)| \xrightarrow{e} |\mathrm{sd}^N(X^{\vee r}\times\triangle^m)|^{\vee r} \xrightarrow{\bigvee_{i=1}^r|f_i|} \bigvee_{i=1}^r|S_\bullet|X^{\vee n_i}|| \xrightarrow{\bigvee_{i=1}^r\epsilon_{X^{\vee n_i}}} \bigvee_{i=1}^r|X^{\vee n_i}|
\end{aligned}$$

where

- N is the integer $\max(N_{f_1},\ldots,N_{f_n})$.
- and for Y a topological space, the map $\epsilon_Y : |S_\bullet(X^{\vee r})|\to Y$ is the counit of the adjunction between topological spaces and simplicial sets.
- $\delta_{\mathrm{sd}^{N_f}(X\times\triangle^m)}\colon\mathrm{sd}^{N_f}(X\times\triangle^m)\to\mathrm{sd}^{N_f}(X\times\triangle^m)\times\mathrm{sd}^{N_f}(X\times\triangle^m)$ is the diagonal map.
- $\pi_2\colon X\times\triangle^m\to\triangle^m$ is the projection.
- $a\colon|\mathrm{sd}^N(\mathrm{sd}^{N_f}(X\times\triangle^m)\times\mathrm{sd}^{N_f}(\triangle^m))|\to|\mathrm{sd}^N(\mathrm{sd}^{N_f}(X\times\triangle^m)\times\triangle^m)|$ is the map $|\mathrm{sd}^N(\mathrm{id}\times\nu_{\triangle^m}\circ\cdots\circ\nu_{\mathrm{sd}^{N_f-1}\triangle^m})|$.
- $b\colon|\mathrm{sd}^N(S_\bullet|X^{\vee r}|\times\triangle^m)|\to|S_\bullet|X^{\vee r}|\times\triangle^m|$ is a homeomorphism, by Lemma 2.7, which states that there is a homeomorphism $h_Z\colon|\mathrm{sd}(Z)|\to|Z|$ for every simplicial set Z (although this homeomorphism is not necessarily natural for simplicial morphisms $Z\to Z'$).
- $c\colon|S_\bullet|X^{\vee r}|\times\triangle^m|\xrightarrow{c}|X^{\vee r}\times\triangle^m|$ is the composite

$$|S_\bullet|X^{\vee r}|\times\triangle^m| \xrightarrow{p} |S_\bullet|X^{\vee r}||\times|\triangle^m| \xrightarrow{|\epsilon_{X^{\vee r}}|\times\mathrm{id}} |X^{\vee r}|\times|\triangle^m| \xrightarrow{q} |X^{\vee r}\times\triangle^m|$$

  where p and q are isomorphisms as the Kelley product commutes with geometric realisation.
- $d\colon|X^{\vee r}\times\triangle^m|\to|\mathrm{sd}^N(X^{\vee r}\times\triangle^m)|$ is the homeomorphism that exists by Lemma 2.7.
- $e\colon|\mathrm{sd}^N(X^{\vee r}\times\triangle^m)|\to|\mathrm{sd}^N(X^{\vee r}\times\triangle^m)|^{\vee r}$ is a homeomorphism because wedge product commutes with geometric realization.
- $f\colon\bigvee_{i=1}^r|X^{\vee n_i}|\to|X^{\vee n_1+\cdots+n_r}|$ is a homeomorphism, as the wedge product commutes with geometric realisation.

We now start the proof of Lemma 3.21.

Lemma 3.21. Since, by Lemma 3.19, the operad $S_\bullet(\mathsf{CoEnd}_{\mathsf{Top}}(|X|))$ is isomorphic to $Q(X)(r)$, it suffices to construct a zig-zag of weak equivalences

$$\mathsf{CoEnd}(X)\xrightarrow{p}R(X)\xleftarrow{q}Q(X).$$

We define $p(r)$ to be the morphism

$$\mathrm{Map}_{\mathrm{Set}_\triangle}(X,\mathrm{Ex}^\infty(\upsilon_{X^{\vee r}}))\colon\mathrm{Map}_{\mathrm{Set}_\triangle}(X,\mathrm{Ex}^\infty(X^{\vee r}))\to\mathrm{Map}_{\mathrm{Set}_\triangle}(X,\mathrm{Ex}^\infty(S_\bullet|X^{\vee r}|))$$

where $\upsilon_{X^{\vee r}} : X^{\vee r}\to S_\bullet|X^{\vee r}|$ is the unit of the singular chains – geometric realization adjunction. Observe that $\mathrm{Ex}^\infty(\upsilon_{X^{\vee r}})\colon\mathrm{Ex}^\infty(X^{\vee r})\to\mathrm{Ex}^\infty(S_\bullet|X^{\vee r}|)$ is a weak equivalence between fibrant simplicial sets. Hence it is a homotopy equivalence, and the functor $\mathrm{Map}_{\mathrm{Set}_\triangle}(X,-)$ preserves homotopy equivalences. Hence p is a weak equivalence.

It remains to check that it induces a morphism of operads. We check this directly. Note first that $\mathrm{Map}_{\mathrm{Set}_\triangle}(X,\mathrm{Ex}^\infty(\upsilon_{X^{\vee r}}))(f)=\mathrm{Ex}^\infty(\upsilon_{X^{\vee r}})\circ f$. Then observe that $N_{\mathrm{Ex}^\infty(\upsilon_{X^{\vee r}})\circ f}=N_f$ and $\max(N_{\upsilon_{X^{\vee r}})\circ f_1},\ldots,N_{\upsilon_{X^{\vee r}})\circ f_n})=\max(N_{f_1},\ldots,N_{f_n})$. Then observe that the morphism

$$|\mathrm{sd}^N(\mathrm{sd}^{N_f}(X\times\triangle^m)\times\triangle^m)| \xrightarrow{\mathrm{sd}^N((\mathrm{Ex}^\infty(\upsilon_{X^{\vee r}})\circ f,N_f)\times\mathrm{id})} |\mathrm{sd}^N(S_\bullet|X^{\vee r}|\times\triangle^m)|$$

factors as

$$|\operatorname{sd}^N(\operatorname{sd}^{N_f}(X\times\triangle^m)\times\triangle^m)|\xrightarrow{\operatorname{sd}^N((f,N_f)\times\mathrm{id})}|\operatorname{sd}^N(X^{\vee r}\times\triangle^m)|$$
$$\xrightarrow{\operatorname{sd}^N(v_{X^{\vee r}}\times\mathrm{id})}|\operatorname{sd}^N(S_\bullet|X^{\vee r}|\times\triangle^m)|$$

Moreover, having first observed that the following diagram is commutative

$$\begin{array}{ccc}
|\operatorname{sd}^N(X^{\vee r}\times\triangle^m)| & \xrightarrow{|\operatorname{sd}^N(v_{X^{\vee r}}\times\mathrm{id})|} & |\operatorname{sd}^N(S_\bullet|X^{\vee r}|\times\triangle^m)| \\
\downarrow h_{(X^{\vee r}\times\triangle^m)} & & \downarrow h_{S_\bullet|X^{\vee r}|\times\triangle^m} \\
|(X^{\vee r}\times\triangle^m)| & \xrightarrow{|(v_{X^{\vee r}}\times\mathrm{id})|} & |(S_\bullet|X^{\vee r}|\times\triangle^m)|,
\end{array}$$

where $h_Z:|\operatorname{sd}Z|\to|Z|$ is the map that exists by Lemma 2.7, we see that the composite

$$|\operatorname{sd}^N(X^{\vee r}\times\triangle^m)|\xrightarrow{|\operatorname{sd}^N(v_{X^{\vee r}}\times\mathrm{id})}|\operatorname{sd}^N(S_\bullet|X^{\vee r}|\times\triangle^m)|\xrightarrow{b}|S_\bullet|X^{\vee r}|\times\triangle^m|$$
$$\xrightarrow{c}|S_\bullet|X^{\vee r}||\times|\triangle^m|\xrightarrow{|\epsilon_{X^{\vee r}}|\times\mathrm{id}}|X^{\vee r}|\times|\triangle^m|\xrightarrow{d}|X^{\vee r}\times\triangle^m|\xrightarrow{e}|\operatorname{sd}^N(X\times\triangle^m)|^{\vee r}$$

is an isomorphism by the *triangle identities* for the $S_\bullet\text{–}|-|$ adjunction. Explicitly, the (left) triangle identity for an adjunction $L\dashv R$ with unit $\eta: id_X\to R\circ L$ and counit $\epsilon: L\circ R\to id_Y$ states that the natural transformation of functors defined as the composite

$$L\xrightarrow{L\eta}LRL\xrightarrow{\epsilon L}L$$

is the identity transformation. Upon further observing that, for the same reason, the composite

$$|\operatorname{sd}^N(X\times\triangle^m)|^{\vee r}\xrightarrow{\bigvee_{i=1}^r|\mathrm{Ex}^\infty(v_{X^{\vee r}})\circ f_i|}\bigvee_{i=1}^r|S_\bullet|X^{\vee n_i}||\xrightarrow{\bigvee_{i=1}^r\epsilon_{X^{\vee n_i}}}\bigvee_{i=1}^r|X^{\vee n_i}|$$

is exactly the map

$$|\operatorname{sd}^N(X\times\triangle^m)|^{\vee r}\xrightarrow{\bigvee_{i=1}^r|f_i|}\bigvee_{i=1}^r|X^{\vee n_i}|,$$

it becomes obvious that γ commutes with p, and so p is a weak equivalence of operads.

Similarly, we define $q(r)$ to be the morphism

$$\mathrm{Map}_{\mathrm{Set}_\triangle}(X,\mu_{S_\bullet|X^{\vee r}|}):\mathrm{Map}_{\mathrm{Set}_\triangle}(X,S_\bullet|X^{\vee r}|)\to\mathrm{Map}_{\mathrm{Set}_\triangle}(X,\mathrm{Ex}^\infty(S_\bullet|X^{\vee r}|)).$$

This is a weak equivalence of simplicial sets for exactly the same reasons that $p(r)$ is. Observe that $N_{q(r)(f)}=0$ for all $f\in Q(X)(r)$. It follows from the form of the operad maps that the morphism q identifies $Q(X)$ with a suboperad of $R(X)(r)$. In particular, q is a morphism of operads, and so a weak equivalence of operads. □

Finally, we can prove the main result of this section.

Theorem 3.16. Let $\Sigma^n X$ be the n–fold suspension of a simplicial set X. As $|\Sigma X|$ is CW-complex, it is in Top. Suspensions are a particular kind of finite limits, and the geometric realization functor commutes with finite limits, so suspensions commute with geometric realization (alternatively, see [5]) and thus that $|\Sigma^n X|$ is a coalgebra over the little n–discs operad in Top. This coalgebra structure is an an operadic morphism $\Phi:\mathbb{D}_n\to\mathrm{CoEnd}_{\mathsf{Top}}(|\Sigma^n X|)$. As discussed above, we can use $S_\bullet$ to transfer these operads and this algebra structure to the category of simplicial sets, producing the following morphism of operads

$$S_\bullet(\Phi):S_\bullet(\mathbb{D}_n)\to S_\bullet(\mathrm{CoEnd}_{\mathsf{Top}}(|\Sigma^n X|))$$

Lemma 3.21 tells us that there is a weak equivalence between $\mathsf{CoEnd}(X)$ and $S_\bullet(\mathrm{CoEnd}_{\mathsf{Top}}(|\Sigma^n X|))$. Observe that in each arity $\mathsf{CoEnd}(X)(n)$ is a mapping space where the target is a Kan complex, hence Kan itself and a fibrant operad in the operadic model structure. By its construction, in each arity $S_\bullet \operatorname{CoEnd}_{\mathsf{Top}}(|\Sigma^n X|)$ is a singular complex and thus as an operad it is also fibrant.

Since we have a weak equivalence between fibrant operads, over the cofibrant replacement $(S_\bullet \mathbb{D}_n)_\infty$ of $S_\bullet \mathbb{D}_n$ we have an induced bijection between the homotopy classes of morphisms of operads

$$[(S_\bullet \mathbb{D}_n)_\infty, \operatorname{CoEnd}_{\mathrm{Set}_\triangle}(\Sigma^n X)] \cong [(S_\bullet \mathbb{D}_n)_\infty, S_\bullet \operatorname{CoEnd}_{\mathsf{Top}}(|\Sigma^n X|)].$$

So we can choose a morphism $\phi : (S_\bullet \mathbb{D}_n)_\infty \to \operatorname{CoEnd}_{\mathrm{Set}_\triangle}(\Sigma^n X)$, such that ϕ is homotopy equivalent to $S_\bullet \Phi$.

Finally to prove that n-fold suspensions are E_n-algebras it suffices to note that all topological operads are fibrant and so the weak equivalence between the little n-discs operad and the geometric realization of the Barratt-Eccles E_n-operad remains one when taking the $S_\bullet$ functor. The Barratt-Eccles E_n-operad $\Gamma^{(n)}$ is weakly equivalent to $S_\bullet|\Gamma^{(n)}|$, and in particular, $(S_\bullet \mathbb{D}_n)_\infty$ can be taken to be the Boardman-Vogt resolution of $\Gamma^{(n)}$; the operad $W(\triangle^1, \Gamma^{(n)})$ which we shall compute next. □

4 Examples of the Berger-Moerdijk W-construction

This section describes what the W-construction looks like in practice in simplicial sets. We start by discussing *associahedra* (sometimes called *Stasheff polytopes*); an infinite sequence of topological polytopes. These were originally discovered by Dov Tamari in his unpublished 1951 PhD thesis [16] (but later independently rediscovered by Jim Stasheff [15]), and significantly predate the study of model categories. Nonetheless the arity n component of the Boardman-Vogt resolution of the associative operad in topological spaces takes the form of a disjoint union of associahedra. We shall informally describe the calculation of these associahedra in low arities before stating some features of the general case. This section is primarily for intuition, as we shall then move on to studying Boardman-Vogt resolution of the associative operad in **simplicial sets**. Here, we shall do the low arity calculations by hand, before giving a complete combinatorial description. What we end up with is a simplicial set that can be realized geometrically as an associahedron (though we do not prove this). Finally, we shall treat the W-construction applied to the Barratt-Eccles operad.

4.0.1 Topological associahedra

Recall that in Top the unit is just the one point set $*$ and that the topological associative operad is defined by

$$\operatorname{Assoc}(n) := \coprod_{\mathbb{S}_n} *.$$

In other words, it is just a discrete collection of points equipped with the free action of $\mathbb{S}_n$. As we are working in topological spaces, it is immediately apparent that this is $\mathbb{S}$-cofibrant and well-pointed. In Top one can verify that $[0,1]$, equipped with the maximum operation

$$\max : [0,1] \otimes [0,1] \to [0,1],$$

satisfies the axioms defining an interval. Thus $W([0,1], \operatorname{Assoc})$ will be a cofibrant replacement for Assoc.

Assoc is a reduced operad so by Remark 2.28 we do not need to consider trees with unary vertices. Also, $\operatorname{Assoc}(0) = \emptyset$, so if T has a vertex with no incoming edges $\overline{\operatorname{Assoc}} = \emptyset$, so we can ignore these trees as well. So when $n \geq 2$, the trees with n inputs that have the most internal edges are the binary trees with $n-2$ internal edges. Every other possible tree can be obtained from these via contraction.

Explicitly when $n = 1, 2$, the largest tree with n inputs has no internal edges so $W([0,1], \operatorname{Assoc})(1)$ is a point and $W([0,1], \operatorname{Assoc})(2)$ is a pair of points. The symmetric group acts on these in the obvious way.

When $n = 3$, we have three trees to consider. Two of these are binary trees and we also have t_3. The two binary trees have 1 internal edge and thus produce intervals. The tree t_3 which produces a point which is identified with 0 of both intervals. Thus it glues both intervals together. In fact, there will be 6 such intervals and the action of the symmetric group is just to permute them.

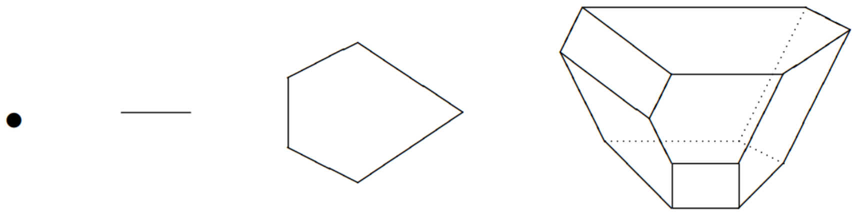


Figure 3: The small associahedra (reproduced from [10])

When $n = 4$ we have 5 binary trees with 2 internal vertices. These all produce $[0,1]^2$. These connect along 5 trees with 1 internal vertices, which means that these glue along a lines. Finally these glue along a single tree t_4 with no internal vertices. The final result is pentagon. Of course, there are $4! = 24$ such pentagons and the group $\mathbb{S}_4$ acts by permuting them.

In higher dimension $W([0,1], \mathrm{Assoc})(n)$ will consist of $n!$ disjoint $n-2$ dimensional polytopes. These will each have

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

vertices, as this is the number of binary trees with n vertices. This polytope is known as the n^{th} associahedra $\mathcal{K}_n$. Figure 3 displays $\mathcal{K}_n$ for $n = 2,3,4,5$.

To describe the operad structure on $W([0,1], \mathrm{Assoc})$, select a point x within the polytope $\mathcal{K}_n$ and one point x_i each in the polytopes $\mathcal{K}_{k_i}$ for $k_i \in \mathbb{N}$ and $0 \leq i \leq n$. Then x has coordinates $(j_1, \dots j_{n-2})$ the $(n-2)$-cube $H(T)$ for some binary tree T and the x_i will have coordinates $(j_{i,1}, \dots j_{i,k_i-2})$ the $(n-2)$-cube $H(T_i)$ for some binary tree T_i with k_i leaves. Consider the tree $S = \gamma_{\mathbb{T}}(T, T_1, \dots, T_n)$. Then let y is the point

$$(j_1, \dots j_{n-2}, j_{1,1}, \dots j_{1,k_i-2}, \dots j_{n,k_n-2}, 1, \dots, 1) \in H(S)$$

Here we have adopted the convention that the first n coordinates correspond to the internal edges of S that come from T, the next k_1 are those coming from T_1 and so on. The final n points come from the newly created n internal edges of S which occur where the T_i have been grafted onto T. We have set all of these values equal to 1. Finally we recall that $H(S)$ has a canonical embedding into $\mathcal{K}_{k_1+\dots+k_n}$. We thus have that $\gamma(x, x_1, \dots, x_n)$ is the image of y under this embedding.

4.0.2 Simplicial associahedra

In the category of simplicial sets, $\triangle^0$ is the initial object and the monoidal product is $\times$. Our first step shall be to show that $\triangle^1$ is an interval in $\mathrm{Set}_\triangle$.

Lemma 4.1. *The standard* 1*-simplex is an interval in* $\mathrm{Set}_\triangle$.

Proof. There is a diagram of the form

$$\triangle^0 \sqcup \triangle^0 \hookrightarrow \triangle^1 \xrightarrow{\sim} \triangle^0.$$

The weak equivalence is just the terminal morphism. The cofibration is given by the pair $(0,1)$. The map 0 corresponds to the morphism given on nondegenerate simplices by

$$\{0\} \longmapsto \{0\}$$

and the map 1 corresponds to

$$\{0\} \longmapsto \{1\}.$$

The morphism $\vee : \triangle^1 \times \triangle^1 \to \triangle^1$ is the maximum operation (ie. the operation induced by the maximum operation on the set $[1]$). It is easy to check that that diagrams appearing in definition 2.15 commute. □

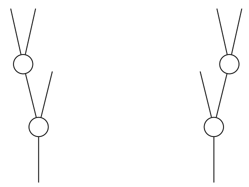


Figure 4: The arity 3 trees with one internal edge

Definition 4.2. The *associative operad in simplicial sets* is given in arity n by

$$\mathrm{Assoc}(n) := \coprod_{\mathbb{S}_n} \triangle^0.$$

equipped with the free action of $\mathbb{S}_n$.

The very low arity calculations are trivial.

Example 4.3. In arity 0,1 and 2, there are no trees to worry about other than the trivial one, the corolla. It follows immediately that

$$W(\triangle^1, \mathrm{Assoc})(n) = W_0(\triangle^1, \mathrm{Assoc})(n) = \coprod_{\mathbb{S}_n} \triangle^0.$$

for $n = 0, 1, 2$.

Remark 4.4. One can easily check that

$$H(T) \times \overline{\mathrm{Assoc}}(T) \times_{Aut(T)} I[S_n] \cong H(T) \times \mathrm{Assoc}(n).$$

This immediately implies that

$$W(\triangle^1, \mathrm{Assoc})(n) = \mathcal{K}_n \times \mathrm{Assoc}(n)$$

for some simplicial set $\mathcal{K}_n$. In other words $W(\triangle^1, \mathrm{Assoc})(n)$ consists of $n!$ copies of the *simplicial associahedron* $\mathcal{K}_n$. We have not yet seen any fully worked out examples of the W-construction. Therefore in the next two examples, we shall ignore this 'obvious' simplification in the interests of gaining fluency with the computations involved. We shall return to it in the proof of Theorem 4.11.

Example 4.5. The first nontrivial case is $n = 3$. To begin with we have

$$W_0(\triangle^1, \mathrm{Assoc})(3) = \coprod_{\mathbb{S}_3} \triangle^0.$$

This comes equipped with a map

$$\alpha_{t_3} : (H(t_3) \times \overline{\mathrm{Assoc}}(t_3)) \times_{Aut(t_3)} I[\mathbb{S}_3] \to W_0(\triangle^1, \mathrm{Assoc})(3)$$

We observe that the domain and codomain of α_{t_3} are Assoc(3) and we recall from the previous section that α_{t_3} is defined to be the identity on Assoc(3). To compute $W_1(\triangle^1, \mathrm{Assoc})(3)$ we must consider the trees with one internal edge and three input edges. These are the two binary trees which are illustrated in figure 4. We shall first study the tree on the left, which we shall denote T. Because there is only one internal edge, we have that $H(T) = \triangle^1$ and $H(T)^- = \triangle^0$. The cofibration

$$H(T)^- \hookrightarrow H(T)$$

given by the map $0 : \triangle^0 \to \triangle^1$. There is a partial operad composition map

$$\circ_1 : \mathrm{Assoc}(2) \times \mathrm{Assoc}(2) \to \mathrm{Assoc}(3).$$

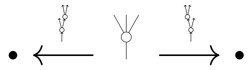


Figure 5: The simplicial set $W(\triangle^1, \text{Assoc})(3)$

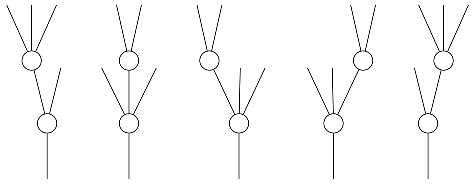


Figure 6: The arity 4 trees with one internal edge

Combining these we obtain

$$\alpha_T^- : (H(T) \times \overline{\text{Assoc}}(T))^- \times_{\text{Aut}(T)} I[\mathbb{S}_n] \to W_0(\triangle^1, \text{Assoc})(3)$$

as $id_{\triangle^0} \times \circ_1$. We also have a map

$$r_T : (H(T) \times \overline{\text{Assoc}}(T))^- \times_{\text{Aut}(T)} I[\mathbb{S}_n] \to (H(T) \times \overline{\text{Assoc}}(T)) \times_{\text{Aut}(T)} I[\mathbb{S}_n]$$

induced by the 0 map. We can repeat this analysis for the right tree S in Figure 4. The only difference is that the partial composition operation is now $\circ_2$. To finish calculating $W_1(\triangle^1, \text{Assoc})(3)$ we now must calculate the pushout appearing in Equation (6). The result is the simplicial set appearing in Figure 5 tensored by $\bigsqcup_{\mathbb{S}_3} \triangle^0$. The simplicial set in the figure consists of three 0-simplices, one of which is associated with the 3-corolla. There are two nondegenerate 1-simplices, one associated to S and the other to T. These have the property that if we apply d_0 to either of them, the result is the 0-simplex associated to t_3. The result of tensoring by $\bigsqcup_{\mathbb{S}_3} \triangle^0$ is that we have 6 disjoint copies of this simplex, each labelled with an element of $\mathbb{S}_3$.

Recall that the pushout also gives us maps

$$\alpha_T : (H(T) \times \overline{\text{Assoc}}(T)) \times_{\text{Aut}(T)} I[\mathbb{S}_n] \to W_0(\triangle^1, \text{Assoc})(3).$$

One observe that $(H(T) \times \overline{\text{Assoc}}(T)) \times_{\text{Aut}(T)} I[\mathbb{S}_n]$ is simply 6 copies of $H(T)$ indexed by $\mathbb{S}_3$. The map α_T consists of mapping each copy of $H(T)$ to 1-simplex on the right via the identity map in the appropriate (meaning labelled with the same element of $\mathbb{S}_3$) copy of Figure 5. There are no larger trees with three input vertices, and thus we conclude that $W_1(\triangle^1, \text{Assoc})(3)$ is also $W(\triangle^1, \text{Assoc})(3)$.

Example 4.6. Moving on to the $n = 4$ case, we have as before that

$$W_0(\triangle^1, \text{Assoc})(4) = \bigsqcup_{\mathbb{S}_4} \triangle^0.$$

This comes equipped with the identity map

$$\alpha_{t_4} : (H(t_4) \times \overline{\text{Assoc}}(t_4)) \times_{Aut(t_4)} I[\mathbb{S}_4] \to W_0(\triangle^1, \text{Assoc})(4).$$

There are five trees with four input vertices and one internal edge. These are shown in Figure 6. We can repeat the analysis appearing in the case $n = 3$ to build $W_1(\triangle^1, \text{Assoc})(4)$. This results in 4! copies of Figure 7, with these copies indexed by $\mathbb{S}_4$.

Finally, as we can see from Figure 8, there are five trees with four input edges and two internal edges. We shall study the tree T on the far left of the figure. We have labelled its internal edges by a and b. We shall denote the tree obtained by collapsing a by T_a (resp. b by T_b). We have that $H(T) = (\triangle^1)_a \times (\triangle^1)_b$ where here $(\triangle^1)_i$ means that this is the component of $H(T)$ associated to the internal edge x. $H(T_a) \cong H_a(T) = \triangle^1_b$ and $H(T_b) \cong H_b(T) = \triangle^1_a$ embed into $H(T)$ in the obvious way. Finally $H(t_4) = \triangle^0$ embeds into $H(T), H(T_a)$ and $H(T_b)$ via

$$\{0\} \mapsto \{0\}.$$

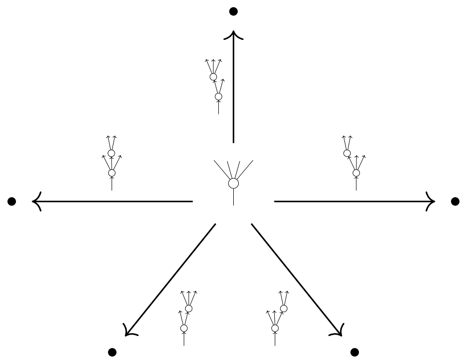


Figure 7: $W_1(\Delta^1, \text{Assoc})(4)$

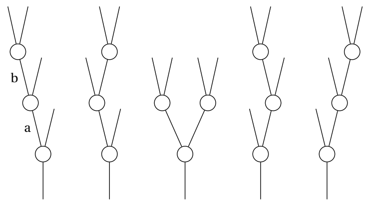


Figure 8: The arity 4 trees with two internal edges

The morphism

$$id_{\text{Assoc}(2)} \times \circ_1 : \text{Assoc}(2) \times \text{Assoc}(2) \times \text{Assoc}(2) \to \text{Assoc}(2) \times \text{Assoc}(3)$$

is the map $\overline{\text{Assoc}}(T) \to \overline{\text{Assoc}}(T_a)$ and $\circ_1 \times id_{\text{Assoc}(2)}$ is the map $\overline{\text{Assoc}}(T) \to \overline{\text{Assoc}}(T_b)$. These maps together describe the map

$$\alpha_T^- : ((\Delta_a^1 \sqcup \Delta_b^1)/\Delta^0) \times \text{Assoc}(4) \to W_1(\Delta^1, \text{Assoc})(4).$$

We also have the obvious embedding

$$r_T : ((\Delta_a^1 \sqcup \Delta_b^1)/\Delta^0) \times \text{Assoc}(4) \hookrightarrow \Delta_a^1 \times \Delta_b^1 \times \text{Assoc}(4).$$

Taking the pushout, we obtain the part of $W_2(\Delta^1, \text{Assoc})(3)$ corresponding to T, as illustrated in Figure 9.

Doing this for all the trees in Figure 8 we obtain the Figure 10. In the diagram we have placed the label for each $H(T)$ at the terminal vertex (see Definition 4.10). There are no trees of arity 4 with more than two internal edges, so we conclude that $W(\Delta^1, \text{Assoc})(4)$ is 24 copies of Diagram 10, each indexed with an element of $\mathbb{S}_4$.

Definition 4.7. A directed graph $\mathcal{G}$ is called *transitively closed* it satisfies the condition that

$$(a,b),(b,c) \in E(\mathcal{G}) \implies (a,c) \in E(\mathcal{G}),$$

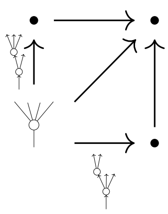


Figure 9: Part of $W_2(\Delta^1, \text{Assoc})(3)$

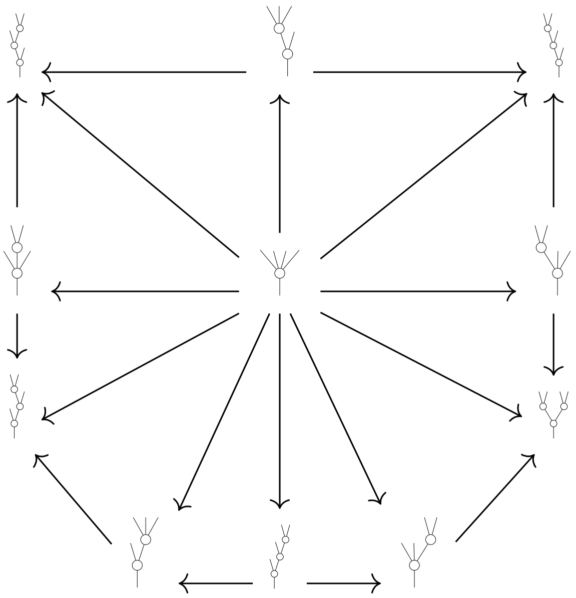


Figure 10: $W(\triangle^1, \text{Assoc})(4)$

where $E(G)$ is the edge set of the graph. In other words, if there is a directed edge from vertices a to b and another from vertices b to c, then there is a edge from a to c.

A transitively closed graph can be regarded as a category.

Definition 4.8. Let $\mathscr{G}$ be a transitively closed graph. We define a category $\overline{\mathscr{G}}$ with objects given by the vertices of $\mathscr{G}$ and with

$$\text{Hom}(a,b) = \begin{cases} \{id\} & \text{if } a = b. \\ \emptyset & \text{if } a \neq b \text{ and } (a,b) \notin E(\mathscr{G}). \\ \{(a,b)\} & \text{if } a \neq b \text{ and } (a,b) \in E(\mathscr{G}). \end{cases}$$

The composite of the morphisms (a,b) and (b,c) is (a,c).

Before describing $W(\triangle^1, \text{Assoc})(n)$, it will be necessary to understand the combinatorial structure of $(\triangle^1)^{\times n}$.

Lemma 4.9. *Let $\mathscr{I}$ be a directed graph on 2^n vertices, each labelled by a sequence of length n with elements in $\{0,1\}$, and such there exists an edge $v_{(\sigma_1,\sigma_2,\ldots,\sigma_n)}$ to $v_{(\tau_1,\tau_2,\ldots,\tau_n)}$ if and only if $\sigma_i \leq \tau_i$ for all i. Then this graph is transitively closed and*

$$(\triangle^1)^{\times n} = \mathscr{N}(\overline{\mathscr{I}})$$

Proof. Each 0-simplex of $(\triangle^1)^{\times n}$ is of the form $\sigma_1 \times \sigma_2 \times \cdots \times \sigma_n$ where $\sigma_i \in [1]$. There is a 1-simplex $\rho \in (\triangle^1)^{\times n}$ such that $d_0(\rho) = \sigma_1 \times \sigma_2 \times \cdots \times \sigma_n$ and $d_1(\rho) = \tau_1 \times \tau_2 \times \cdots \times \tau_n$ if and only if $\sigma_i \leq \tau_i$ for all i. For $m \geq 3$ the non-degenerate simplices of $(\triangle^1)^{\times n}$ are

$$((\triangle^1)^{\times n}_m)^{nd} = \{s_{(q_1)}(\theta_1) \times s_{(q_2)}(\theta_2) \times \cdots \times s_{(q_n)}(\theta_n) : m > q_i \geq 0 \text{ and } q_i \neq q_j \text{ when } i \neq j\}$$

where θ_i is the 1-simplex in the i^{th} copy of $\triangle^1$ in the product, where $s_{(k)} = s_{n-1}s_{n-2}\cdots\widehat{s_k}\cdots s_0$ and we use the notation $\widehat{x}$ to mean that we omit x. The nondegenerate simplices of $\mathscr{N}(\overline{\mathscr{I}})_m$ will be the simplices of the form

$$(\sigma_{1,1} \times \cdots \times \sigma_{1,n}) \circ \cdots \circ (\sigma_{m,1} \times \cdots \times \sigma_{m,n})$$

where

- $\sigma_{i,j} \in \triangle^1_1$.

- for each $i \in [1,m]$, there exists a unique $p(i) \in [1,n]$ such that $\sigma_{i,p(i)} = \mathrm{id}$.
- fix $j \in [1,n]$ and suppose that there exists $q(j) \in [1,n]$ such that $\sigma_{q(j),j} = \mathrm{id}$. Then for all $i > q(j)$, $\sigma_{i,j} = s_0(1)$ and for all $i < q(j)$, $\sigma_{i,j} = s_0(0)$.

Then there is a function

$$f : \mathcal{N}(\overline{\mathcal{I}})^{ng}_m \to ((\Delta^1)^{\times n}_m)^{ng}$$

$$(\sigma_{1,1} \times \cdots \times \sigma_{1,n}) \circ \cdots \circ (\sigma_{m,1} \times \cdots \times \sigma_{m,n}) \mapsto s_{(p(1)-1)}(\mathrm{id}) \times \cdots \times s_{(p(n)-1)}(\mathrm{id})$$

One can check that this extends to an isomorphism of simplicial sets between $\mathcal{N}(\overline{\mathcal{I}})$ and $(\Delta^1)^{\times n}$. □

Definition 4.10. The *terminal vertex* of $(\Delta^1)^{\times n}$ is the vertex $v_{(1,1,\ldots,1)}$.

Theorem 4.11. *Let $\mathcal{G}$ be the directed graph on*

$$\frac{3P_{n-1}(3) - P_{n-2}(3)}{4n}$$

vertices, where P_n is the n^{th} Legendre polynomial and where each vertex v_T is labelled by an n-ary non-unital tree. The edges of $\mathcal{G}$ are defined as follows; there is a directed edge from v_T to v_S if and only there exists $D \subseteq E(S)$ such that $T = S/D$. Then this graph is transitively closed and

$$W(\Delta^1, \mathrm{Assoc})(n) = \bigsqcup_{\mathbb{S}_n} \mathcal{N}(\overline{\mathcal{G}})$$

where the action of $\mathbb{S}_n$ given by permutation of components of the disjoint union.

Remark 4.12. Recall that the Legendre polynomials $P_n(x)$ are a system of complete and orthogonal polynomials, indexed by the positive integers and defined inductively by setting $P_1(x) = 1$ and requiring that $\int_{-1}^{1} P_n(x)P_m(x)dx = 0$ for all $m < n$. The sequence

$$a_n = \frac{3P_{n-1}(3) - P_{n-2}(3)}{4n}$$

is known as the little Schroeder numbers (OEIS sequence A001003 [9]). In particular a_n counts the number of non-unital n-ary trees, for a proof see [14].

Proof. First, we make the observation that $H(T) \times \overline{\mathrm{Assoc}}(T)) \times_{\mathrm{Aut}(T)} I[\mathbb{S}_n]$ is isomorphic to $H(T) \times \bigsqcup_{\mathbb{S}_n} \Delta^0$. The simplicial set $W(\Delta^1, \mathrm{Assoc})(n)$ therefore has the form

$$\bigsqcup_{\mathbb{S}_n} \mathcal{K}_n \qquad\qquad (\text{or } \mathcal{K}_n \times \mathrm{Assoc}(n))$$

where $\mathcal{K}_n$ is a simplicial set that remains to be computed. We shall do this by induction. Our inductive hypothesis shall be that $W_i(\Delta^1, \mathrm{Assoc})(n)$ admits the following description.

Let $\mathcal{H}_i$ be a directed graph with vertices indexed by the set of trees with n input edges and i or fewer internal edges, and such that there exists a directed edge from v_T to v_S if and only there exists $D \subseteq E(S)$ such that $T = S/D$. Then

$$W_i(\Delta^1, \mathrm{Assoc})(n) = \bigsqcup_{\mathbb{S}_n} \mathcal{N}(\overline{\mathcal{H}_i})$$

Moreover we assume, letting $\mathcal{I}$ be as in Lemma 4.9, that the map α_T is $\mathcal{N}(\overline{\beta_T})$ where

$$\beta_T : \mathcal{I} \to \mathcal{H}_i$$

is the map which sends the vertex in $\mathcal{I}$ indexed by the binary sequence $(\sigma_1, \ldots \sigma_k)$ to the vertex in $\mathcal{H}_i$ indexed by the tree T/F where $F = \{e_i \in E(T) : \sigma_i = 0\}$.

Firstly, when $i = 0$, we have that $W_0(\Delta^1, \mathrm{Assoc})(n) = \mathrm{Assoc}(n)$. So our hypothesis holds in the base case.

Secondly, recall that the following diagram, which is a specialization of 6, is a pushout

$$\begin{array}{ccc} \coprod_{[T],T\in\mathbb{T}(n,k+1)} H^-(T)\times \text{Assoc}(n) & \xrightarrow{\coprod \alpha_T^-} & W_k(\Delta^1,\text{Assoc})(n) \\ \downarrow & & \downarrow \\ \coprod_{[T],T\in\mathbb{T}(n,k+1)} H(T)\times \text{Assoc}(n) & \xrightarrow{\coprod \alpha_T} & W_{k+1}(\Delta^1,\text{Assoc})(n) \end{array}$$

Without loss of generality, we can ignore the symmetric action. This is now exactly what we wish to show because for each $T\in\mathbb{T}(n,k+1)$ we have

- The simplicial subset
$$\alpha_T(H(T))\subset W_{k+1}(\Delta^1,\text{Assoc})(n)$$
has exactly one vertex, the terminal one, which is not in $W_k(\Delta^1,\text{Assoc})(n)$. We define this to be the vertex v_T.
- Let $\sigma\in W_{k+1}(\Delta^1,\text{Assoc})(n)$ be a 1-simplex. Then $d_1(\sigma)=v_T$ if and only if $d_0(\sigma)\in\alpha_T(H(T))$, that is, if $d_0(\sigma)$ is indexed by a tree S such that there exists $D\in E(T)$ such that $S=T/D$.
- By Lemma 4.9, $\alpha_T(H(T))\cong(\Delta^1)^{\times k+1}$ is equal to $\mathcal{N}(\overline{\mathcal{I}})$. One can show that the nerve functor preserves small coproducts in the category of small categories. So
$$W_{k+1}(\Delta^1,\text{Assoc})(n)=\mathcal{N}(\mathcal{H}_{k+1})$$
as desired.

Therefore, our inductive hypothesis holds for $i=k+1$.

Finally, as noted in Remark 2.28, the construction of $W(\Delta^1,\text{Assoc})(n)$ relies only on n-ary trees with no unitary vertices. Therefore our induction terminates once we reach $W_{n-2}(\Delta^1,\text{Assoc})(n)$. □

4.1 The cofibrant version of the Barratt-Eccles E_n-operad

This subsection is dedicated to giving a concrete description of the Boardman-Vogt resolution of the Barratt-Eccles E_n-operad. The 0-simplices of the Barratt-Eccles operad are the same as those of the associative operad, so some of our analysis in the last subsection carries over. The new feature is that we now have non-degenerate simplices in dimension greater than 0. These behave in a more complicated fashion. More precisely, as we saw above, each tree in $\mathbb{T}$ with k internal vertices is associated to $n!$ copies of the k-cube $H(T)$ in the associahedron, indexed by $\mathbb{S}_n$. We are also to show that there is a subset K_T of $\mathbb{S}_n$ associated to T. Every i-simplex σ in $\Gamma^{(k)}(n)$ has $i+1$ vertices, which are elements of $\mathbb{S}_n$. If all these vertices are in K_T, then $\sigma\times H(T)$ is a simplex in $W(\Delta^1,\Gamma^{(k)})(n)$. Let us begin.

Definition 4.13. Let T be a tree with n input edges and k internal edges. Let T' be a tree given by collapsing one of its internal edges d. Then there is a group homomorphism

$$f_d: Aut(T)\to Aut(T')$$

given by partial composition in the associative operad at the collapsed edge d. Since $Aut(t_n)=\mathbb{S}_n$, iterating this procedure until we arrive at the n-corolla induces a map

$$F_T: Aut(T)\to\mathbb{S}_n.$$

One can easily check that this map is independent of the order in which we collapse the internal edges. The image of F_T will be a subset of $\mathbb{S}_n$ which we call the *subset associated to* T.

Example 4.14. Consider the left tree in Figure 4. We know that $Aut(T)=\mathbb{S}_2\times\mathbb{S}_2$ and the map F_T is given by the map $\circ_1:\mathbb{S}_2\times\mathbb{S}_2\to\mathbb{S}_3$. The subset of $\mathbb{S}_3$ associated to T will be the permutations

$$\{e,(1,2),(1,2,3),(1,3)\}$$

Similarly the relevant map for the right tree in the figure would be $\circ_2$. The subset of $\mathbb{S}_3$ associated to this tree will be the permutations

$$\{e,(1,3),(1,3,2),(2,3)\}$$

We shall can now give our description of the Boardman-Vogt resolution of the Barratt-Eccles E_n-operad $W(\triangle^1, \Gamma^{(n)})$.

Notation 4.15. Consider $W(\triangle^1, \mathrm{Assoc})$ as described in the previous section. Recall that, for each $T \in \mathbb{T}(r)$ there is a map

$$\alpha_T : H(T) \times \overline{\mathrm{Assoc}}(T)) \times_{\mathrm{Aut}(T)} I[\mathbb{S}_r] \to W_r(\triangle^1, \mathrm{Assoc})(r) \hookrightarrow W(\triangle^1, \mathrm{Assoc})(r)..$$

Further recall that $H(T) \times \overline{\mathrm{Assoc}}(T)) \times_{\mathrm{Aut}(T)} I[\mathbb{S}_r] \cong \mathbb{S}_r$. Under this identification, for each $\sigma \in \mathbb{S}_r$ we define

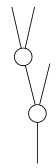
$$H(T)_\sigma := \alpha_T(H(T) \times \sigma) \subset W(\triangle^1, \mathrm{Assoc})(r).$$

Theorem 4.16. *Let $n > 0$ be an integer. The Boardman-Vogt resolution of the Barratt-Eccles E_n-operad $W(\triangle^1, \Gamma^{(n)})$ admits the following complete description.*

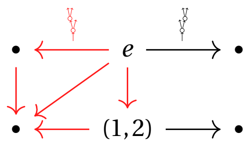
- *The simplicial set $W(\triangle^1, \Gamma^{(n)})$ has a simplicial subset isomorphic to $W(\triangle^1, \mathrm{Assoc})$. This subset contains the entire 0-skeleton of $W(\triangle^1, \Gamma^{(n)})$.*
- *Let $\sigma = (\sigma_1, \dots, \sigma_k) \in \Gamma^{(n)}$. Let $T \in \mathbb{T}$ such that $\sigma_i \in K_T$, for $0 \le i \le k$. Then $G_T^\sigma = \sigma \times H(T)$ is a simplicial subset of $W(\triangle^1, \Gamma^{(n)})$ such that $\sigma_i \times H(T) = H(T)_{\sigma_i}$ for $0 \le i \le k$.*
- *Every simplex of $W_i(\triangle^1, \Gamma^{(n)})(r)$ is in either $W(\triangle^1, \mathrm{Assoc})$ or one of the G_T^σ.*

Before we prove this result, we shall briefly illustrate what it means in practice.

Example 4.17. Consider the simplex $\sigma = (e, (1,2)) \in \Gamma(3)_1$. The permutations e and $(1,2)$ are in K_T where T is the tree



The simplicial subset G_T^σ is therefore the red part of the following diagram.



where the top line is the associahedron with index e and the second line is the associahedron with index $(1,2)$.

Proof. By Remark 2.28, the construction of $W(\triangle^1, \Gamma^{(n)})(r)$ relies only on r-ary trees with no unitary vertices. We shall use induction on the number i of internal edges of these trees. To be precise, we shall assume that $W_i(\triangle^1, \Gamma^{(n)})(r)$ admits the following description.

- The simplicial set $W_i(\triangle^1, \Gamma^{(n)})(r)$ has a simplicial subset isomorphic to $W_i(\triangle^1, \mathrm{Assoc})(r)$. This subset contains the entire 0-skeleton of $W_i(\triangle^1, \Gamma^{(n)})(r)$.
- Let $\sigma = (\sigma_1, \dots, \sigma_k) \in \Gamma^{(n)}$. Let $T \in \mathbb{T}(r, i)$ such that $\sigma_j \in K_T$, for $0 \le j \le k$. Then $G_T^\sigma = H(T) \times \sigma$ is a simplicial subset of $W(\triangle^1, \Gamma^{(n)})$ such that $H(T) \times \sigma_j = H(T)_{\sigma_j}$ for $0 \le j \le k$. Moreover, the map α_T is induced by the identity map $H(T) \times \sigma \to G_T^\sigma$ for all $\sigma \in \overline{\Gamma^{(n)}}(T)$.
- Every simplex of $W(\triangle^1, \Gamma^{(n)})$ is in either $W(\triangle^1, \mathrm{Assoc})$ or one of the G_T^σ.

First observe that

$$W_0(\triangle^1, \Gamma)(r) := \Gamma(r).$$

Since $K_{t_r} = \mathbb{S}_r$, our hypothesis is true when $i = 0$. Next, suppose that it is true when $i = k$. Then

$$\begin{array}{ccc} \bigsqcup_{[T], T\in\mathbb{T}(r,k+1)} (H \times \Gamma^{(n)})^{-}(T) \times_{\mathrm{Aut}(T)} I[\mathbb{S}_r] & \xrightarrow{\sqcup \alpha_T^{-}} & W_k(\Delta^1, \Gamma^{(n)})(r) \\ \downarrow & & \downarrow \\ \bigsqcup_{[T], T\in\mathbb{T}(r,k+1)} (H(T) \times \overline{\Gamma^{(n)}}(T)) \times_{\mathrm{Aut}(T)} I[\mathbb{S}_r] & \xrightarrow{\sqcup \alpha_T} & W_{k+1}(\Delta^1, \Gamma^{(n)})(r) \end{array}$$

Let $\sigma = (\sigma_0, \dots, \sigma_k) \in \Gamma^{(n)}$. Then one can easily show that $\sigma \in \overline{\Gamma^{(n)}}(T)$ if and only if $\sigma_i \in K_T$ for $0 \le i \le k$. Therefore $\bigsqcup_{[T], T\in\mathbb{T}(r,k+1)} (H(T) \times \overline{\Gamma^{(n)}}(T)) \times_{\mathrm{Aut}(T)} I[\mathbb{S}_r]$ is the union of two kinds of simplicial sets.

- We have a $(k-1)$-cube $H(T) \times (\sigma_0)$ for each tree $T \in \mathbb{T}(n, k+1)$ and each 0-simplex $\sigma_0 \in \Gamma^{(n)}$.
- We have a simplicial set $H(T) \times \sigma$ for each $\sigma \in \Gamma^{(n)}$ such that the vertices of σ lie in K_T.

By our inductive hypothesis, the simplicial subset

$$\left(\bigsqcup_{[T], T\in\mathbb{T}(r,k+1)} \alpha_T \right) \left(\bigcup_{\sigma_0 \in \mathbb{S}_r} H(T) \times (\sigma_0) \right) \subseteq W_{k+1}(\Delta^1, \Gamma^{(n)})(r)$$

will be isomorphic to $W_{k+1}(\Delta^1, \mathrm{Assoc})(r)$. The second type of simplicial subset are exactly the subsets of type 2. □

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